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EQUIVARIANT DIFFERENTIAL TOPOLOGY†

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INTRODUCTION

THE AIM of this paper is to establish the basic propositions of differential topology (as presented in Milnor [9], for example) for G -manifolds where G is a compact Lie group.

Mostow [11] and Palais [12] proved that any compact G -manifold can be imbedded in a Euclidean G -space. In §1 a technique of de Rham's [5] is used to prove an analogue of the Whitney Imbedding Theorem, namely that any G -manifold M^n , "subordinate" to the representation V , can be imbedded in V^{2n+1} .

Section 2 concerns the classification of G -vector bundles. The precise statement is: The equivalence classes of k -dimensional G -vector bundles over M^n "subordinate" to V are in a natural one-to-one correspondence with the equivariant homotopy classes of maps of M into $G_k(V^t)$, the grassmannian of k -planes in V^t , if $t > n + k$. The existence of a classifying map is proved via a transversality argument. The equivalence of bundles induced by homotopic maps can be shown to follow from the existence and uniqueness of solution curves of vector fields. Atiyah [1] has proved a similar theorem for compact topological spaces.

Section 3 develops a cobordism theory for G -manifolds. Equivariant homotopy groups are defined and it is shown that the unoriented cobordism group of G -manifolds of dimension n , subordinate to V are isomorphic to the equivariant homotopy classes of maps of the sphere in $V^{2n+3} \oplus \mathbf{R}$ into the Thom space of the universal bundle over $G_k(V^{2n+3} \oplus \mathbf{R})$ where $k + n = (2n + 3)$ dimension of V , if G is abelian or finite. There is a severe technical difficulty in establishing even a weak transversality theorem for G -manifolds; hence, the existence of the isomorphism for arbitrary compact Lie groups is still an open question.

Section 4 generalizes the results of R. Palais [14] on Morse Theory on Hilbert Manifolds to the case of G -manifolds. It is shown that "Morse functions" are dense in the set of *invariant* real valued functions on M if M is finite dimensional. Also it is shown that passing a critical value of a Morse function corresponds to adding on "handle-bundles" over orbits or more generally over non-degenerate critical submanifolds. Morse inequalities are then deduced for the case of critical submanifolds. The results in this section were announced in [15]. Some of the results in this section have been obtained independently by Meyer [6].

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§0. NOTATIONS AND DEFINITIONS

Let G be a compact Lie group and X a completely regular topological space. An action of G on X is a continuous map $\psi: G \times X \rightarrow X$ such that $\psi(e, x) = x$ and $\psi(g_1 g_2, x) = \psi(g_1, \psi(g_2, x))$ for all $x \in X$ and $g_1, g_2 \in G$. The pair (X, ψ) will be called a G -space. We will denote by $\bar{g}: X \rightarrow X$ the map given by $\bar{g}(x) = \psi(g, x)$ and $\psi(g, x)$ will be shortened to gx . X_G will denote $\{x \in X | gx = x \text{ for all } g \in G\}$; G_x is the isotropy group, $\{g \in G | gx = x\}$. If Y is another G -space and $f: X \rightarrow Y$ then f is equivariant if, for all $g \in G$, $f \circ \bar{g} = \bar{g} \circ f$ and invariant if $f \circ \bar{g} = f$. If $\mathcal{M}(X, Y)$ is some set of maps of X into Y (differentiable, linear, etc.) then G acts on $\mathcal{M}(X, Y)$ by $gf = \bar{g} \cdot f \cdot \bar{g}^{-1}$. Clearly $\mathcal{M}(X, Y)_G$ is the set of equivariant maps in $\mathcal{M}(X, Y)$. If $H \subset G$ is a closed subgroup $X|H$ will denote the pair $(X, \psi|X \times H)$.

Let M be a C^∞ Hilbert manifold [7] with or without boundary. M will be called a G -manifold if the action $\psi: G \times M \rightarrow M$ is a differentiable map. The tangent bundle $T(M)$ of a G -manifold M is also a G -manifold with the action $gX = d\bar{g}_p(X)$ for $X \in T(M)_p$. More generally, if $\pi: E \rightarrow B$ is a fibre bundle and each $\bar{g}: E \rightarrow E$ is a bundle map then π will be called a G -bundle; if, in addition, π is a differentiable fibre bundle and E and B are G -manifolds then π is a differentiable G -bundle. If the G -vector bundle $\pi: E \rightarrow B$ has a Riemannian metric, $\langle \cdot, \cdot \rangle$, and \bar{g} is an isometry for each $g \in G$ then π is called a Riemannian G -vector bundle. If E is a Riemannian G -vector bundle then $\|e\| = \langle e, e \rangle^{1/2}$, $E(r) = \{e \in E | \|e\| \leq r\}$, $\dot{E}(r) = \{e \in E | \|e\| < r\}$ and $\check{E}(r) = \{e \in E | \|e\| = r\}$. We write $\dot{E} = \dot{E}(1)$ and $\check{E} = \check{E}(1)$. Note that $T(M) \rightarrow M$ is a differentiable G -vector bundle; if $T(M) \rightarrow M$ is a Riemannian G -vector bundle then M is a Riemannian G -space. A Riemannian G -vector bundle V over a point is an (orthogonal) representation. V^t will denote the t -fold direct sum of V with itself.

If M is a G -manifold and $\Sigma \subset M$ is a compact invariant submanifold then $\pi: \nu(\Sigma) \rightarrow \Sigma$ the normal bundle of Σ is a differentiable G -vector bundle; moreover, by a theorem of Koszul [6], there is an equivariant diffeomorphism $\nu(\Sigma) \rightarrow U$ where U is an open neighborhood of Σ in M . In particular, if $x \in M$, $B_x(r)$ will denote the image of $\nu(Gx)(r)$ under some such diffeomorphism, $S_x(r)$ will denote the image of $\pi^{-1}(x)(r)$. We write $B(x) = B_x(1)$, $S(x) = S_x(1)$. $B_x(r)$ is a tubular neighborhood of Gx and $S_x(r)$ is a slice at x .

If V is a representation of G then $G_k(V)$ will denote the grassmanian of k -planes in V . $G_k(V)$ may be thought of as orthogonal projections on V with nullity k ; hence G acts on $G_k(V) \subset \mathcal{M}(V, V)$ and $G_k(V)$ is a G -manifold with this action. Denote by $\mu_k(V)$ the universal bundle over $G_k(V)$; the fibre at $P \in G_k(V)$ is the null space of P . The inner product on V induces a metric on $\mu_k(V)$ and with this metric $\mu_k(V) \rightarrow G_k(V)$ is a Riemannian G -vector bundle. Let $W \subset V$ be an invariant subspace of dimension k . For each $P \in G_k(V)$ we have a representation of G_P on the null space of P ; in particular, for $P \in G_k(V)_G$ we have a representation of G and if Q and P are in the same component of $G_k(V)_G$ the representations at P and Q are equivalent. Hence, we denote by $G_W(V)$ the set of k -planes $G_k(V)_G$ which are equivalent to W . Clearly $G_W(V)$ is a component of $G_k(V)_G$. We write $\mu_W(V)$ for $\mu_k(V)|_{G_W(V)}$.

If $f: X \rightarrow V$ is any map into a Euclidean G -space then averaging f over the group means an equivariant map f^* defined by $f^*(x) = \int_G g^{-1}f(gx)dg$ or the invariant map \bar{f} defined by $\bar{f}(x) = \int_G f(gx)dg$ as the context dictates.

Let X be an equivariant vector field on M , i.e., $X_{gp} = gX_p$. If $\sigma_p(t)$ denotes the maximal solution curve to X with initial condition p then by the equivariance of X , $g\sigma_p(t)$ and $\sigma_{gp}(t)$ are both solution curves with initial condition gp and, hence, by uniqueness of solution curves $g\sigma_p(t) = \sigma_{gp}(t)$. Therefore the flow generated by X is equivariant. If $f: M \rightarrow \mathbf{R}$ is an invariant function on the Riemannian G -space M then f gives rise to the vector field gradient of f , ∇f , by $\langle \nabla f_p, X \rangle = df_p(X)$. Note that $\langle g\nabla f_p, X \rangle = \langle \nabla f_p, g^{-1}X \rangle = df_p(g^{-1}X) = d(f \cdot g^{-1})_{gp}(X) = df_{gp}(X) = \langle \nabla f_{gp}, X \rangle$ for all $X \in T(M)_{gp}$ so $g\nabla f_p = \nabla f_{gp}$ and hence ∇f is an equivariant vector field.

$C_G(M, N)$ will denote the equivariant C_∞ maps between the finite dimensional G -manifolds M and N with the C^k topology for some fixed k . If $f \in C_G(M, N)$, $\varepsilon > 0$ and $\psi: R^n \rightarrow M$, $\varphi: R^n \rightarrow N$ are coordinate charts for M and N respectively then a sub-base for the neighborhoods of f in the C^k topology is given by $\{h \in C_G(M, N) | N_k(\varphi^{-1} \cdot f \cdot \psi - \varphi^{-1} \cdot h \cdot \psi)(x) < \varepsilon \text{ for } \|x\| \leq 1\}$ where $N_k(w)(x) = \sum_{j=0}^k \|d^j w_x\|$, $w: R^m \rightarrow R^n$ and $\|\cdot\|$ denotes the usual norm on multilinear transformations. $C_G(M, N)$ is a space of the second category.

§1. GENERALIZED WHITNEY THEOREM

In this section we prove an analogue of the Whitney imbedding theorem for G -manifolds. Let V be a finite dimensional orthogonal representation of G .

PROPOSITION 1.1. *If M^n can be immersed in V^t then M can be immersed in V^{2n} .*

Proof. Let $f: M \rightarrow V^t$ be an immersion and let W be a k -dimensional irreducible representation of G contained in V . It will be sufficient to show that if W occurs s times in V^t and $s > 2n$, then there is an equivariant projection $P: V^t \rightarrow V^t$ with null space isomorphic to W such that $P \cdot f$ is an immersion.

To that end consider the diagram $\dot{T}(M) \xrightarrow{\widetilde{df}} \dot{V}^t \xleftarrow{i} \dot{\mu}_W(V^t) \xrightarrow{\pi} G_W(V^t)$ where $\widetilde{df}(X) = df(X)/\|df(X)\|$, $i(P, w) = w$ and $\pi(P, w) = P$. The pair (P, w) represents a point in $\dot{\mu}_k(V^t)$ as a projection with null space isomorphic to W and a unit vector in that null space. Since W is irreducible, i is a differentiable homeomorphism into. To show that i is an imbedding we let $X_{(P, w)}$ be any tangent vector at (P, w) and let $\lambda(t) \in \dot{V}^t$, $\gamma(t) \in G_W(V^t)$ be curves such that $(\gamma'(0), \lambda'(0)) = X_{(P, w)}$. Then $di_{(P, w)}X = \lambda'(0)$; but if $\lambda'(0) = 0$, $\gamma'(0) = d\pi\lambda'(0) = 0$ since $\gamma(t) = \pi \circ \lambda(t)$. Hence $di(X) = 0$ implies $X = 0$ and so i is an imbedding.

Since the dimension of $\dot{T}(M) = 2n - 1$, $\dim df(\dot{T}(M)) \cap i(\dot{\mu}_W(V^t)) \leq 2n - 1$ and since i is an imbedding and π is differentiable the $\dim \pi \circ i^{-1}(\widetilde{df}(\dot{T}(M)) \cap i(\dot{\mu}_W(V^t))) \leq 2n - 1$. But the \dim of $G_W(V^t)$ is $(s - 1)l$ where l is the dimension of the division algebra $\text{Hom}(W, W)_G$. Hence, if $(s - 1)l > 2n - 1$, and in particular if $s > 2n$ there is a projection P such that $P \circ \widetilde{df}(w) = 0$ if and only if $w = 0$, i.e., $P \circ f$ is an immersion. Moreover, if $P_0 \in G_W(V^t)$, P can be chosen arbitrarily close to P_0 .

Continuing in this fashion, we eventually find a projection T , the composition $\cdots P_4 \cdot P_3 \circ P_2 \circ P_1$, such that $T \circ f$ is an immersion and the range of T is isomorphic to V^{2n} .

PROPOSITION 1.2. *If M^n admits a 1-1 immersion in V^t , then M can be 1-1 immersed in V^{2n+1} .*

Proof. Let $f: M \rightarrow V^t$ be a 1-1 immersion and consider the diagram $M \times M \xrightarrow{\Delta} \dot{V}^t \xleftarrow{i} \dot{\mu}_W(V^t) \xrightarrow{\pi} G_W(V^t)$ where $\alpha(x, y) = f(x) - f(y) / \|f(x) - f(y)\|$. Since $\dim(M \times M) = 2n$, $\dim \pi \circ i^{-1}[\alpha(M \times M - \Delta) \cap i(\dot{\mu}_W(V^t))] \leq 2n$ and hence if $t > 2n + 1$ we can find a projection $P: V^t \rightarrow V^t$ with null space isomorphic to W such that $i(\pi^{-1}(P))$ is disjoint from the image of df (so that $P \circ f$ is an immersion) and from the image of α . If $P(f(x)) = P(f(y))$ then $P \circ \alpha(x, y) = 0$ and hence $\alpha(x, y) \in i(\pi^{-1}(P))$; thus $P \circ f$ is 1-1.

COROLLARY 1.3. *Suppose that M admits an immersion, f , in V^t . Then any map $g: M \rightarrow V^{2n}$ can be C^k -approximated by an immersion. The approximation is also uniform.*

Proof. The approximation, \bar{g} , will be of the form $\bar{g}(x) = g(x) + Af(x)$ where A is a bounded linear map: $V^t \rightarrow V^{2n}$ and $\|A\| < \varepsilon$. By a diffeomorphism of V^t we may assume $\|f(x)\| < 1$ for all x and hence \bar{g} will be a uniform approximation. To make \bar{g} a C^k approximation on some compact set C , we need only replace $f(x)$ by $\delta f(x)$ where $\delta = \varepsilon / \sup_{x \in C} N_k(f(x))$ (see §0). Let i_1 (resp. i_2) denote the inclusion of V^t (resp. V^{2n}) in $V^t \times V^{2n}$ and let P_0 denote the internal projection of $V^t \times V^{2n}$ onto the second factor. Applying Prop. 1.1 to the map $f \times g: M \rightarrow V^t \times V^{2n}$ yields a projection P such that $P \circ (f \times g)$ is an immersion and $\|P - P_0\| < \varepsilon$. If $E = P(V^t \times V^{2n})$, then $P \circ i_2$ is an isomorphism onto E for ε sufficiently small and thus $(P \circ i_2)^{-1}: E \rightarrow V^{2n}$ is defined. Let $\bar{g} = (P \circ i_2)^{-1} \circ P \circ (f \times g)$. Note that \bar{g} is an immersion and $\bar{g}(x) = g(x) + (P \circ i_2)^{-1} \circ P \circ (f(x), 0) = g(x) + (P \circ i_2)^{-1} \circ P \circ i_1(f(x)) = g(x) + Af(x)$.

COROLLARY 1.4. *Suppose that M admits a one-to-one immersion, f , in V^t . Then any map $g: M \rightarrow V^{2n+1}$ can be C^k -approximated by a one-to-one immersion. The approximation is also uniform.*

Proof. Essentially the same as above.

COROLLARY 1.5. *If M admits a one-to-one immersion in V^t then M can be imbedded as a closed subset of V^{2n+1} .*

Proof. Let $g: M \rightarrow V^{2n+1}$ be a proper map and apply the previous corollary. To get a proper map, let $\bar{\psi}_i$ be a locally finite partition of unity with compact support and average over the group to get ψ_i , an invariant partition of unity. Let $f: M \rightarrow V^{2n+1}$ be a one-to-one immersion (Cor. 1.4). If $f(y) = 0$ (there is at most one such point), let ψ_1, \dots, ψ_r denote those functions with $y \in \text{support } \psi_i$ and let $m_i = \inf_{\psi_i(x) > 0} \|f(x)\|$ $i > r$. Then define

$$g(x) = \sum_{i=r+1}^{\infty} i\psi_i(x)f(x)/m_i.$$

Since $g^{-1}([0, n]) \subset \bigcup_{i=1}^n \text{support } \psi_i = \text{compact set}$ for $n > r$, g is proper.

Remark. If the origin is not in the image of f in Props. 1.1, 1.2, 1.3, 1.4, 1.5, then the new map can be chosen so as to avoid the origin also. If $\beta: M \rightarrow \dot{V}^t$ is defined by $\beta(x) = f(x)/\|f(x)\|$ then the dimension of the image of β is less than n , choose the projection, P , in Props. 1.1, 1.2 so as to avoid the n -dimensional set $\pi \circ i^{-1}(\beta(M) \cap i(\dot{\mu}_W(V^t)))$. With such a choice of P the conclusion follows in Cors. 1.3, 1.4, and 1.5.

Definition. Let V be a finite dimensional orthogonal representation of G . A G -manifold M is said to be subordinate to V if for each $x \in M$ there exists an invariant neighborhood U of x and an equivariant differentiable imbedding of U in $V^t - \{0\}$ for some t . $\mathcal{G}(V)$ is the category whose objects are G -manifolds subordinate to V and whose maps are continuous equivariant maps.

PROPOSITION 1.6. *There are only a finite number of orbit types in $\mathcal{G}(V)$.*

Proof. Let Ω be an orbit type in $\mathcal{G}(V)$ and $x \in \Omega$. By assumption there is a differentiable imbedding of an invariant neighborhood of x in V^t for some t . Hence there is a one-to-one equivariant immersion of Ω in V^{2n+1} where $n = \dim \Omega$; in particular since Ω is compact Ω can be imbedded in $V^{2 \dim G + 1}$. But $V^{2 \dim G + 1}$ contains only a finite number of orbit types [13]

PROPOSITION 1.7. *M^n is in $\mathcal{G}(V)$ if and only if (i) for each $m \in M$, G/G_m is one of the orbit types in $\mathcal{G}(V)$ and (ii) there is a G_m equivariant monomorphism*

$$T(M)_m / T(G_m)_m \rightarrow V^n.$$

Proof. Necessity is clear and sufficiency follows from 1.7.10 of [13].

COROLLARY 1.8. *M^n is in $\mathcal{G}(V)$ if and only if M is locally imbeddable in $V^{n+2 \dim G + 1} - \{0\}$.*

PROPOSITION 1.9. *If M is in $\mathcal{G}(V)$ then M can be imbedded in V^t for some t .*

Proof. By Cor. 1.8 we may cover M by the interiors of compact invariant sets U_α such that each U_α admits an imbedding $f_\alpha: U_\alpha \rightarrow V^s - \{0\}$ where $s = 2 \dim G + \dim M + 1$. Since M is paracompact and has dimension n there is a countable refinement of U_α by compact invariant sets U_{ij} $i = 0, 1, \dots, n; j \in \mathbb{Z}^+$, such that $U_{ij} \cap U_{ik} = \emptyset$ if $j \neq k$ [8]. Let $f_{ij}: U_{ij} \rightarrow V^s - \{0\}$ be an imbedding; let r_j be a diffeomorphism of the positive reals onto $(j, j+i)$ and let

$$\tilde{f}_{ij}(x) = r_j(\|f_{ij}(x)\|) \frac{f_{ij}(x)}{\|f_{ij}(x)\|}.$$

Then each \tilde{f}_{ij} is an imbedding and the images of f_{ij}, f_{ik} are disjoint if $j \neq k$; hence the map $f_i: U_i = \bigcup_{j=1}^{\infty} U_{ij} \rightarrow V^s - \{0\}$ given by $f_i(x) = \tilde{f}_{ij}(x)$, $x \in U_{ij}$, is an imbedding. Let \tilde{f}_i imbed U_i in the unit sphere in V^{2s} by $\tilde{f}_i(x) = (r_0(\|f_i(x)\|)f_i(x)/\|f_i(x)\|, \sqrt{1 - r_0(\|f_i(x)\|)^2}f_i(x)/\|f_i(x)\|)$. Finally, let $h_i: M \rightarrow I$ be differentiable invariant functions with support $h_i \subset U_i$ and such that $\bigcup_{i=0}^n \text{Int } h_i^{-1}(1)$ covers M and define $f: M \rightarrow V^{2(n+1)s}$ by $f(x) = (h_0(x)\tilde{f}_0(x), h_1(x), \tilde{f}_1(x), \dots, h_n(x)\tilde{f}_n(x))$. f is clearly equivariant and differentiable. If $x \in \text{Int } h_i^{-1}(1)$, $\pi_i \circ df = d\tilde{f}_i$ and hence f is an immersion; if $f(x) = f(y)$ then $h_i(y) = 1$ and $\tilde{f}_i(y) = \tilde{f}_i(x)$ and so $x = y$ and \tilde{f} is 1-1. If $\{f(x_n)\} \rightarrow f(x)$ then $\{h_i(x_n)\} \rightarrow h_i(x) = 1$ and hence $x_n \in U_i$ for n large and since $\{h_i(x_n)\} \rightarrow 1$, $\{\tilde{f}_i(x_n)\} \rightarrow \tilde{f}_i(x)$ but since \tilde{f}_i is an imbedding $\{x_n\} \rightarrow x$ and hence f is an imbedding.

COROLLARY 1.10. (Generalized Whitney Theorem). *If M is in $\mathcal{G}(V)$ then any map $f: M \rightarrow V^t$ can be approximated C^k and uniformly by an equivariant immersion if $t \geq 2n$ and by an equivariant 1-1 immersion if $t \geq 2n + 1$. Moreover, if C is a closed subset of M and $f|_C$ is an immersion (1-1 immersion), the approximation \tilde{f} may be chosen to agree with f on C .*

Proof. The first statement follows from Prop. 1.9 and Cors. 1.3, 1.4. To prove the last statement let $g: M \rightarrow V^l$ be an imbedding with $\|g(x)\| = 1$. Let $h: M \rightarrow I$ be an invariant function such that $C \subset \text{Int } h^{-1}(1)$ and $f|_{\text{support } h}$ is an immersion (1-1 immersion). Then $x \rightarrow (f(x), (1 - h(x))g(x))$ is an immersion (1-1 immersion) of M in $V^l \times V^l$. The approximation of Cor. 1.3 (Cor. 1.4) has the desired properties.

Remark. If V is a representation of G in a Hilbert space then by the Peter-Weyl theorem, V can be decomposed $V = \bigoplus_{i=1}^{\infty} V_i^{r_i}$ where the V_i are finite dimensional irreducible representations of G , $0 \leq r_i \leq \infty$ and the direct sum is in the Hilbert sense. Let $V^* = \bigoplus_{i=1}^{\infty} V_i$. Then by Prop. 1.7 and the fact that closed subgroups of G obey the descending chain condition we see that M is in $\mathcal{G}(V)$ if and only if M is in $\mathcal{G}(V^*)$. In addition, all propositions of this section except 1.6 hold for $\mathcal{G}(V^*)$ and hence for $\mathcal{G}(V)$. As a consequence of this remark we have that any equivariant differentiable map $M^n \rightarrow L^2(G)^{2n+1}$ can be approximated by an equivariant 1-1 immersion since $L^2(G)$ contains at least one copy of each irreducible representation of G . In particular, if $f: M \rightarrow \mathbf{R} \subset L^2(G)^{2n+1}$ is proper, say $f(x) = \sum i\psi_i(x)$ where ψ_i is an equivariant partition of unity, the approximation will be an imbedding. Hence

COROLLARY 1.11. *Any G -manifold M^n can be imbedded as a closed subset of $L^2(G)^{2n+1}$ and hence has a complete invariant metric.*

COROLLARY 1.12. *If $f: M \rightarrow N^n$ is a continuous equivariant map then f can be approximated by a differentiable map.*

Proof. By Cor. 1.11, N may be considered as a retract of an open invariant neighborhood U of $N \subset L^2(G)^{2n+1}$ with retraction $r: U \rightarrow N$. Let $f_1: M \rightarrow U$ be a differentiable approximation to f ([9]) and average f_1 over the group to get f^* . The approximation is given by $r \circ f^*$.

§2. CLASSIFICATION OF G -VECTOR BUNDLES

Definition. Let $\pi: E \rightarrow M$ be a G -vector bundle of fibre dimension $k < \infty$ over the G -manifold M . π is said to be subordinate to the representation V of G if, for each $m \in M$, the representation of G_m on $\pi^{-1}(m)$ is equivalent to a subrepresentation of $V^k|_{G_m}$. The category $\mathcal{B}(V)$ will have as objects G -vector bundles subordinate to V and bundle homomorphisms for maps.

Remark. $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$ are the same category where V^* contains exactly one copy of each irreducible representation occurring in V .

If $\pi: E \rightarrow M$ is a G -vector bundle and $f: N \rightarrow M$ is equivariant then $f^*\pi \subset N \times E$ inherits a natural G -structure from the product which makes $f^*\pi \rightarrow N$ a G -vector bundle. Moreover, if π is in $\mathcal{B}(V)$ then so is $f^*\pi$. In particular, $\pi: \mu_k(V^l) \rightarrow G_k(V^l)$ is in $\mathcal{B}(V)$ and hence so is $f^*\pi$ for any equivariant map $f: N \rightarrow G_k(V^l)$. The next theorem due to R. Palais shows that "all" bundles over G -manifolds are obtained in this way.

THEOREM 2.1. *Let $\pi: E^{n+k} \rightarrow M^n$ be in $\mathcal{B}(V)$ and let $f: E|C \rightarrow \mu_k(V')$ be a bundle map where $C \subset M$ is a closed invariant subspace. If $t \geq n + k$, then f can be extended to a bundle map $h: E \rightarrow \mu_k(V')$.*

Proof. Consider the G -vector bundle $\text{Hom}(E, V')$ over M with fibre $\text{Hom}(\pi^{-1}(m), V')$ at m . The action of G is given by $gT = \bar{g} \cdot T \cdot \bar{g}^{-1}$ where $T \in \text{Hom}(\pi^{-1}(m), V')$ and $gT \in \text{Hom}(\pi^{-1}(gm), V')$. A section s of $(\text{Hom}(E, V'))$ is said to be non-singular if $s(m)$ is a non-singular linear transformation for each $m \in M$.

LEMMA 2.2. *There is a natural equivalence $\theta: \text{non-singular sections of } \text{Hom}(E, V') \rightarrow \text{bundle maps of } E \text{ into } \mu_k(V')$. Under this equivalence, equivariant sections correspond to equivariant bundle maps.*

Proof. Almost a tautology. If s is a non-singular section of $\text{Hom}(E, V')$ then $s(m)(\pi^{-1}(m))$ is a k -plane in V' and if $e \in \pi^{-1}(m)$ then $s(m)(e)$ is a point in that k -plane. Hence s defines a bundle map $\theta(s): E \rightarrow \mu_k(V')$. Moreover, if s is equivariant, $s(gm)(ge) = \bar{g} \cdot s(m) \cdot \bar{g}^{-1}(ge) = g \cdot s(m)(e)$, hence $\theta(s)$ is equivariant. Similarly, if $f: E \rightarrow \mu_k(V')$ is a bundle map then $x \rightarrow f|\pi^{-1}(x)$ defines a non-singular section of $\text{Hom}(E, V')$ which is equivariant if f is equivariant.

Let $\Gamma_G(E)$ denote the G equivariant sections of $\text{Hom}(E, V')$ with the C^0 topology and let $\mathcal{N}_G(A, M) \subset \Gamma_G(E)$ denote those sections which are non-singular at points of $A \subset M$. Note that $\Gamma_G(E)$ is of the second category.

LEMMA 2.3. *$\mathcal{N}_G(M, M)$ is dense in $\Gamma_G(E)$ if $t \geq n + k$.*

Proof. Note that $\mathcal{N}_G(A, M)$ is open in $\Gamma_G(E)$ if A is compact; hence, by Baire's theorem, it is sufficient to find a countable number of compact sets C_i such that $\cup C_i = M$ and $\mathcal{N}_G(C_i, M)$ is dense in $\Gamma_G(E)$ and hence $\cap \mathcal{N}_G(C_i, M) = \mathcal{N}_G(\cup C_i, M)$ is dense in $\Gamma_G(E)$.

By the induction metatheorem of [13], we may assume the lemma true for all proper closed subgroups of G ; in particular, if $x \in M - M_G$ we may assume that $\mathcal{N}_{G_x}(S_x, S_x)$ is dense in $\Gamma_{G_x}(E|S_x)$ where S_x is a slice at x . Moreover, the restriction map $\rho: \Gamma_G(E) \rightarrow \Gamma_{G_x}(E|S_x)$ is open and hence $\rho^{-1}(\mathcal{N}_{G_x}(S_x, S_x)) = \mathcal{N}_G(GS_x, M)$ is open and dense in $\Gamma_G(E)$.

Now let $y \in M_G$, U a neighborhood of y in M_G , and let v_1, \dots, v_k be sections of $E|U$ such that $v_1(y), \dots, v_k(y)$ spans $\pi^{-1}(y) = F$. Let $T: U \times F \rightarrow \pi^{-1}(U)$ by $T(u, \sum a_i v_i(y)) = \sum a_i v_i(u)$; averaging over the group yields an equivariant homomorphism $T^*: U \times F \rightarrow \pi^{-1}(U)$ which is an isomorphism at y and hence in some compact neighborhood $B(y)$ of y ; i.e., $E|B(y)$ is equivariantly isomorphic to $B(y) \times F$. Thus $\Gamma_G(E|B(y))$ is homeomorphic to $C^0(B(y), \text{Hom}_G(F, V'))$. Let $N_j = \{T \in \text{Hom}_G(F, V') | \text{rank } T = j\}$; N_j is a disjoint union of submanifolds of $\text{Hom}_G(F, V')$ and each component has codimension at least $t - j$ and hence codimension greater than n for $j < k$. Since $\mathcal{N}_G(B(y), B(y))$ consists of those sections which are transverse regular to $\bigcup_{j < k} N_j$, i.e., avoid $\bigcup_{j < k} N_j$, $\mathcal{N}_G(B(y), B(y))$ is open and dense in $\Gamma_G(E)$.

Since the restriction map $\rho: \Gamma_G(E) \rightarrow \Gamma_G(E|B(y))$ is open $\mathcal{N}_G(B(y), M)$ is open and dense in $\Gamma_G(M)$. Covering M_G by a countable number of sets $B(y_i)$ and $M - M_G$ by a countable number of sets GS_{x_i} , the lemma follows.

Now let $\Gamma_G(E, s_0) \subset \Gamma_G(E)$ denote those sections which extend s_0 . $\Gamma_G(E, s_0)$ is non-empty since any extension of s_0 may be averaged over the group to get an equivariant extension; moreover, $\Gamma_G(E, s_0)$ is of the second category. If $A \subset M$ is compact and $A \cap C = \emptyset$ then $\rho: \Gamma_G(E, s_0) \rightarrow \Gamma_G(E|A)$ is open hence $\mathcal{N}_G(A, M) \cap \Gamma_G(E, s_0)$ is dense in $\Gamma_G(E, s_0)$. Covering $M - C$ by a countable number of compact sets $B(y_i)$, GS_{x_i} with $B(y_i) \cap C = \emptyset$, $GS_{x_i} \cap C = \emptyset$ we have $\mathcal{N}_G(M, M) \cap \Gamma_G(E, s_0)$ is dense in $\Gamma_G(E, s_0)$.

Remark. See [1] for a quick proof of the following theorem when M is compact.

THEOREM 2.4. *Let $\pi: E \rightarrow M \times I$ be a differentiable G -vector bundle. Then there is an equivariant bundle equivalence $(E|M \times 0) \times I \rightarrow E$.*

Proof. We may assume that the structural group of E has been reduced to $O(k)$. Let $\pi: P \rightarrow M \times I$ be the principal bundle of E . P is a G -bundle with compact fibre. It is clearly sufficient to show that there is an equivariant bundle equivalence $(P|M \times 0) \times I \rightarrow P$. To that end let X^* be an invariant vector field on P projecting onto d/dt , i.e., $X_p^* = gX_p^*$ and $d\pi(X_p) = d/dt|_{\pi(p)}$. We may obtain such a vector field directly using an equivariant partition of unity or alternatively define $X^* = \text{grad}(p_2 \circ \pi)$ where $p_2: M \times I \rightarrow I$ is the projection and the gradient is defined with respect to some invariant Riemannian metric for $T(P)$. Next let

$$X_p = \int_{O(k)} d\gamma^{-1} X_{\gamma p}^* d\gamma.$$

Since the actions of $O(k)$ and G on P commute and since $\pi(\gamma p) = \pi(p)$ we have that X is a G equivariant and an $O(k)$ equivariant vector field on P projecting onto d/dt . Let $\sigma_p(t)$ denote the unique maximal solution curve to the vector field X with initial condition p . By the G -equivariance of X we have that $g\sigma_p(t) = \sigma_{gp}(t)$. Let $U \subset P|(M \times 0) \times I$ be the maximum domain of the equivariant map $\theta: U \rightarrow P$ given by $\theta(p, t) = \sigma_p(t)$. We wish to show that $U = P|(M \times 0) \times I$. But if $p \in P|M \times 0$, $\pi\sigma_p(t) = (m, t)$ and hence $\sigma_p(t) \in \pi^{-1}(m \times t)$ for all $(m, t) \in U$ since $d\pi(X) = d/dt$. Hence, to determine the domain of σ_p we need only consider the bundle $\pi^{-1}(m \times I) \rightarrow m \times I$. But $\pi^{-1}(m \times I)$ is compact and hence σ_p is defined for all $t \in I$. Thus $U = (P|M \times 0) \times I$. Since X is an $O(k)$ invariant vector field, θ is a bundle map. Hence θ is an equivariant bundle equivalence.

COROLLARY 2.5. *If $\pi: E \rightarrow M$ is a differentiable G -vector bundle and $f, g: N \rightarrow M$ are homotopic then $f^*\pi$ is equivalent to $g^*\pi$.*

Proof. Let $h: N \times I \rightarrow M$ be the homotopy. Let $U \subset M \times M$ be an invariant neighborhood of the diagonal such that if $(x, y) \in U$ then there exists a unique minimal geodesic γ_{xy} with $\gamma_{xy}(0) = x$ and $\gamma_{xy}(1) = y$. Let $p: U \times I \rightarrow M$ by $p(x, y, t) = \gamma_{xy}(t)$. Let $\bar{h}: N \times I \rightarrow M$ be a differentiable approximation to h such that $w(n, t) = (\bar{h}(n, t), h(n, t)) \in U$ for all $(n, t) \in N \times I$. Then $p_0^*\pi$ is equivalent to $p_1^*\pi$ by Theorem 2.4. Hence $w^*p_0^*\pi = (p_0 \circ w)^*\pi = \bar{h}^*\pi$ is equivalent to $w^*p_1^*\pi = h^*\pi$. But $\bar{h}^*\pi$ is a product by the theorem since \bar{h} is differentiable, hence $h^*\pi$ is a product, i.e., $f^*\pi \approx g^*\pi$.

COROLLARY 2.6. *The equivalence classes of k -dimensional G -vector bundles over M^n subordinate to V are isomorphic to the equivariant homotopy classes of maps of M into $G_k(V)$ if $t \geq n + k + 1$.*

Proof. Follows formally as in [16].

§3. COBORDISM AND EQUIVARIANT HOMOTOPY GROUPS

Let W be a finite dimensional orthogonal representation of G and let $D(W)$ (resp $S(W)$) denote the unit ball (resp unit sphere) in W . If X is a G -space, let X^W denote the space of continuous equivariant maps of $S(W)$ into X with the compact open topology. If $f: X \rightarrow Y$ is equivariant there is an obvious induced map $f^W: X^W \rightarrow Y^W$; the assignment $X \rightarrow X^W$, $f \rightarrow f^W$ is a covariant functor from the category of G -spaces and equivariant maps to the category of topological spaces and continuous maps.

Definition. A G -homotopy triple (X, A, a) is a G -space X , an invariant subspace A , and a fixed point a (i.e. $G_a = G$) in that subspace. If (X, A, a) is a homotopy triple and $n \geq 1$ we define the n th W -homotopy group of (X, A, a) by $\pi_n^W(X, A, a) = \pi_n(X^W, A^W, a^W)$. If $f: (X, A, a) \rightarrow (Y, B, b)$ is an equivariant map of triples the induced homomorphism $f_*: \pi_n^W(X, A, a) \rightarrow \pi_n^W(Y, B, b)$ is defined by $f_*: \pi_n(X^W, A^W, a^W) \rightarrow \pi_n(Y^W, B^W, b^W)$.

If $A = \{a\}$ we denote $\pi_n^W(X, A, a)$ by $\pi_n^W(X, a)$; $\pi_0^W(X, a)$ is defined to be $\pi_0(X^W, a^W)$.

Remark. If G is the trivial group and $W = \mathbb{R}$, then $S(W) = S^0$ and the above definition reduces to $\pi_n^W(X, A, a) = \pi_n(X^W, A^W, a^W) = \pi_n(X \times X, A \times A, (a, a)) \approx \pi_n(X, A, a) \oplus \pi_n(X, A, a)$.

$\pi_n^W(X, A, a)$ may alternatively be defined as equivariant homotopy classes of maps $(D(W \times \mathbb{R}^{n-1}), S(W \times \mathbb{R}^{n-1}), D(\mathbb{R}^{n-1})) \rightarrow (X, A, a)$. In particular $\pi_1^W(X, a)$ is the set of homotopy classes of maps $S(W \times \mathbb{R}) \rightarrow X$ which carry both "north" and "south" poles to a .

If $G_a \neq G$ then $(X|_{G_a}, A|_{G_a}, a)$ is a G_a homotopy triple and one can consider the G_a equivariant homotopy groups $\pi_n^{W'}(X|_{G_a}, A|_{G_a}, a)$ where W' is any representation of G_a (not necessarily of the form $W|_{G_a}$). Note, however, that any G_a equivariant map $W' \rightarrow X$ extends uniquely to a G equivariant map $W' \times_{G_a} G \rightarrow X$ where $W' \times_{G_a} G$ is a G -vector bundle over G/G_a . Moreover, if $\pi: E \rightarrow G/G_a$ is any G -vector bundle over G/G_a such that the representation of G_a on $\pi^{-1}(\{e\})$ is equivalent of W' , the equivalence $W' \rightarrow \pi^{-1}(\{e\})$ extends by equivariance to a G -bundle equivalence $W' \times_{G_a} G \rightarrow E$. Thus, E is determined by the representation of G_a on $\pi^{-1}(\{e\})$. Hence we may define the groups $\bar{\pi}_n^{W'}(X, A, a)$ as G -equivariant homotopy classes of maps $\{D(E \oplus \mathbb{R}^{n-1}), S(E \oplus \mathbb{R}^{n-1}), *\} \rightarrow (X, A, a)$ where $\pi: E \rightarrow G/G_a$ is the unique G -vector bundle with fibre equivalent to W' , $E \oplus \mathbb{R}^{n-1}$ denotes the Whitney sum of E with a trivial bundle of dimension $n-1$ and $* = \{x \in E \oplus \mathbb{R}^{n-1} | x = (0, y) \text{ and } \pi(x) = \{e\} \in G/G_a\}$. Clearly $\bar{\pi}_n^{W'}(X, A, a) = \pi_n^{W'}(X|_{G_a}, A|_{G_a}, a)$.

Let V be a finite dimensional orthogonal representation of G . We wish to develop a cobordism theory for $\mathcal{G}(V)$.

Definition. The compact G -manifolds M_1^n, M_2^n are said to be V -cobordant, $M_1 \sim_V M_2$ (or cobordant, $M_1 \sim M_2$ if no confusion will result), if there exists a compact G -manifold N^{n+1} in $\mathcal{G}(V)$ with ∂N^{n+1} equivariantly diffeomorphic to $M_1 \cup M_2$.

PROPOSITION 3.0. \sim_V is an equivalence relation.

Proof. Symmetry and reflexivity are obvious and transitivity follows from the fact that there is an equivariant diffeomorphism of $\partial N \times [0, 1)$ onto an open neighborhood of ∂N in N .

Definition. $\eta_n(V)$ will denote the unoriented cobordism group of equivalence classes of n -dimensional compact G -manifolds in $\mathcal{G}(V)$. The group operation is given by $[M_1] + [M_2] = [M_1 \cup M_2]$, i.e. disjoint union. Similarly one can consider the oriented cobordism groups $\Omega_n(V)$.

Remark. Appropriate choices of G, V yield the equivariant cobordism groups considered by Conner and Floyd in [3] and [4].

Let $T_k(W)$ denote the Thom space of the bundle $\mu_k(W) \rightarrow G_k(W)$. $T_k(W)$ may be thought of as $\mu_k(W)(\varepsilon)/\dot{\mu}_k(W)(\varepsilon)$; G -acts on $T_k(W)$ in the obvious way and the fixed point $\{\dot{\mu}_k(W)(\varepsilon)\}$ will be denoted by ∞ . We wish to define a homomorphism $\theta: \eta_n(V) \rightarrow \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ where $h \geq n + 3$ and $k + n = (n + h)\dim V$.

Let $[M] \in \eta_n(V)$ and $i: M \rightarrow V^{2n+1} \subset V^{n+h}$ be an imbedding with $0 \notin i(M)$ (cor 1.10). There is a bundle monomorphism $v(M) \rightarrow T(V^{n+h})|i(M) = M \times V^{n+h}$ via the invariant metric on V^{n+h} and hence a bundle map $b: v(M) \rightarrow \mu_k(V^{n+h}) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$. Let $E: T(V^{n+h}) \rightarrow V^{n+h}$ be the end-point map; i.e. $E(v, x) = v + x$ where $x \in V^{n+h}$ and v is a tangent vector at x . Then $\bar{E} = E|v(M)(\delta) \rightarrow V^{n+h}$ is an equivariant diffeomorphism onto a neighborhood U of $i(M)$ for some $\delta > 0$; choose δ small enough so that $0 \notin U$. Let $f_{M,i}: V^{n+h} \rightarrow T_k(V^{n+h} \oplus \mathbf{R})$ be defined by $f_{M,i}|U = q \circ b \circ E^{-1}$, $f_{M,i}(V^{n+h} - U) = \infty$, where $q: \mu_k(V^{n+h} \oplus \mathbf{R}) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})(\varepsilon)/\dot{\mu}_k(V^{n+h} \oplus \mathbf{R})$ is the identification map and $\varepsilon < \delta$. Extending $f_{M,i}$ to the one point compactification of V^{n+h} , i.e. to $S(V^{n+h} \oplus \mathbf{R})$, we get, via the above Thom construction an element $\theta([M]) \in \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$.

PROPOSITION 3.1. θ is a well defined homomorphism.

Proof. Let Q^{n+1} be a compact manifold in $\mathcal{G}(V)$, $\partial Q = M_1 \cup M_2$, and let $i_j: M_j \rightarrow V^{2n+1} - \{0\}$ $j = 1, 2$ be imbeddings. We must show that f_{M_1, i_1} is equivariantly homotopic to f_{M_2, i_2} and hence that $\theta([M])$ is independent of the choice of representative or imbedding.

If $c > 0$ then $ci_1: M \rightarrow V^{2n+1}$ is an imbedding and f_{M_1, ci_1} is clearly homotopic to f_{M_1, i_1} hence we may assume, by choosing c large enough, that $i_1(M_1) \cap i_2(M_2) = \emptyset$. Let $U_j, j = 1, 2$, be an equivariant collaring of M_j in Q , i.e. U_j is an invariant neighborhood of M_j , with equivariant diffeomorphism $\psi_j: M_j \times [0, 2) \rightarrow U_j$ such that $\psi_j|M_j \times \{0\}$ is the identity. Let $i_3: U_1 \cup U_2 \rightarrow V^{n+h} \times [0, 5] \subset V^{n+h} \oplus \mathbf{R}$ by

$$i_3(q) = \begin{cases} (i_1(x), t) & \text{if } q = \psi_1(x, t) \\ (i_2(x), 5 - t) & \text{if } q = \psi_2(x, t) \end{cases}$$

and extend i_3 differentiably to $i_4: Q \rightarrow V^{n+h} \times [0, 5]$ so that $i_4(Q - U_1 \cup U_2) \subset V^{n+h} \times [2, 3]$. If $Q_G \neq \emptyset$ we insist that $i_4|Q_G$ be transverse regular to $\{0\} \times [0, 5]$ in $V_g^{n+h} \times [0, 5]$, i.e. $i_4(Q_G) \cap \{0\} \times [0, 5] = \emptyset$. Then i_4 may be averaged over G to get an equivariant differentiable map $i_5: Q \rightarrow V^{n+h} \times [0, 5]$. Since $h \geq n + 3$, i_5 may be approximated by an equivariant 1-1 immersion (and hence an embedding) $i: Q \rightarrow V^{n+h} \times [0, 5]$ with $i|U_1 \cup U_2 = i_3$ (Corollary 1:10). Note that $i(Q) \cap \{0\} \times [0, 5] = \emptyset$. [If $x \notin Q_g$ this follows since i is an imbedding; for $x \in Q_g$ we note that $i_5(Q_g) \cap \{0\} \times [0, 5] = \emptyset$ and hence for a sufficiently close approximation i , $i(Q_g) \cap \{0\} \times [0, 5] = \emptyset$.] Then we apply the Thom construction as before to get an equivariant homotopy $f_{Q,i}: S(V^{n+h} \oplus \mathbf{R}) \times [0, 5] \rightarrow T_k(V^{n+h} \oplus \mathbf{R})$ with

$f_{Q,i}|S(V^{n+h} \oplus \mathbf{R}) \times \{0\} = f_{M_1,i_1}$ and $f_{Q,i}|S(V^{n+h} \oplus \mathbf{R}) \times \{5\} = f_{M_2,i_2}$. Note that for each $t \in [0, 5]$, $f_{Q,i}|S(V^{n+h} \oplus \mathbf{R}) \times \{t\} : (S(V^{n+h} \oplus \mathbf{R}), \{0, \infty\}) \rightarrow (T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ if the neighborhood U of $i(Q)$ in $V^{n+h} \times [0, 5]$ used in the Thom construction is chosen small enough so that $U \cap 0 \times [0, 5] = \emptyset$. Hence $[f_{M_1,i_1}] = [f_{M_2,i_2}] \in \pi_1^{n+h}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ and thus θ is well defined. Clearly θ is a homomorphism.

If G is trivial, i.e. $G = e$, then it is well known that θ is an isomorphism [9]. One defines a map $\lambda : \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty) \rightarrow \eta_n(v)$ by $\lambda[f] = f_1^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ where (i) f_1 is homotopic to f , (ii) “differentiable,” and (iii) transverse regular (TR) to $G_k(V^{n+h} \oplus \mathbf{R})$. If f_2 is any other such map, then there exists a homotopy $F : S(V^{n+h} \oplus \mathbf{R}) \times [0, 5] \rightarrow T_k(V^{n+h} \oplus \mathbf{R})$ such that $F_0 = f_1$, $F_5 = f_2$ and F is TR to $G_k(V^{n+h} \oplus \mathbf{R})$; hence $F^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ is a cobordism between $f_1^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ and $f_2^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ and λ is well defined. Clearly $\lambda \circ \theta = \text{identity}$. One then shows that λ is a monomorphism by using the fact that $\mu_k(V^{n+h} \oplus \mathbf{R})$ is $(n+1)$ universal (since $G = e$, $V = \mathbf{R}$, and $k = h$).

Serious difficulties arise in trying to carry out this proof when $G \neq e$. First of all, if $f : M \rightarrow N$ is a differentiable equivariant map and $W \subset N$ a compact submanifold, it is not true, in general, that f can be approximated by a map $f_1 : M \rightarrow N$ which is TR to W . For example, let $G = Z_2$, $M = \text{one point}$, $N = \tilde{\mathbf{R}}$ the real line with Z_2 acting by reflection, $W = 0 \in \tilde{\mathbf{R}}$ and $f(x) = 0$, $x \in M$. Clearly f is the only equivariant map $M \rightarrow N$ and is not TR to W .

However, in the special case we are considering, $M = S(V^{n+h} \times \mathbf{R})$, $N = T_k(V^{n+h} \oplus \mathbf{R})$, $W = G_k(V^{n+h} \oplus \mathbf{R})$ one can find in each equivariant homotopy class a map f which is TR to W if G is a “nice” group. However, if f_1 and f_2 are two such maps which are equivariantly homotopic there will not, in general, be an equivariant homotopy h between them satisfying (ii) and (iii). For example, let $G = Z_2$, $V = \mathbf{R} \oplus \tilde{\mathbf{R}}$, $n = 0$, $h = 3$. Let M be a point, $i : M \rightarrow (\mathbf{R} + \tilde{\mathbf{R}})^3$ and consider the maps $f_{M,i}$, $\bar{g} \circ f_{M,i}$ where $e \neq g \in Z_2$; both maps are transverse regular to $G_k(V^3 \oplus \mathbf{R})$ and $f_{M,i}$ is equivariantly homotopic to $\bar{g} \circ f_{M,i}$ but there is no TR homotopy between them as can be shown by a simple determinant argument. In addition $\pi : \mu_k(V^{n+h} \oplus \mathbf{R}) \rightarrow G_k(V^{n+h} \oplus \mathbf{R})$ is not necessarily $(n+1)$ universal. It turns out that the notion of “consistent transverse regularity” (CTR) is sufficient to overcome these difficulties.

The following lemmas are preparatory to proving the transversality theorem.

LEMMA 3.2. *Let M , N be G -manifolds and $f : M \rightarrow N$ a differentiable equivariant map. If C is a closed invariant subspace of M and $h : C \rightarrow N$ is a differentiable equivariant homotopy of $f|_C$ then h can be extended to a differentiable equivariant homotopy of f . Moreover, if U is an open neighborhood of C , the extension F may be chosen so that $F|_{M-U} = f|_{M-U}$.*

Proof. By Proposition 1.66 of [13] and Corollary 1.11 of § 2, N is a G -ANR. Hence, the map $\bar{F} : M \times \{0\} \cup C \times I \rightarrow N$ given by $\bar{F}|_{M \times \{0\}} = f$, $\bar{F}|_{C \times I} = h$ can be extended to a map also called \bar{F} defined in an invariant neighborhood V of $M \times \{0\} \cup C \times I$ in $M \times I$. V contains an open invariant set of the form $U_1 \times I$ where $U_1 \supset C$. Let $\alpha : M \rightarrow I$ be differentiable, invariant with support $\alpha \subset U_1 \cap U$ and $\alpha(C) = 1$. Define $F : M \times I \rightarrow N$ by $F(x, t) = \bar{F}(x, \alpha(x)t)$.

LEMMA 3.3. Let $f: M \rightarrow N$ be differentiable and equivariant and $W \subset N$ a closed invariant submanifold. Let C be a closed subset of M_G and suppose that $f|_{M_G}$ is transverse regular (TR) to W_G in N_G at points of C . Then there exists a homotopy f_t such that

- (i) $f_0 = f$,
- (ii) $f_t|_C = f|_C$ and
- (iii) $f_1|_{M_G}$ is TR to W_G in N_G .

Proof. By the standard transversality lemma (§1.35 of [9]) there exists a homotopy $h_t: M_G \rightarrow N_G$ such that $h_0 = f|_{M_G}$, $h_t|_C = f|_C$ and h_1 is TR to W_G in N_G . Since M_G is a closed subset of M the homotopy h_t may be extended to a differentiable equivariant homotopy f_t of f by Lemma 1.

LEMMA 3.4. Let $f: M \rightarrow N$ be a differentiable equivariant map of G manifolds and let $C \subset U \subset M$ where C is closed and invariant and U is open in M . If $h: U \rightarrow N$ is a differentiable equivariant map with $f|_C = h|_C$ then there is an equivariant homotopy F_t and an open set V with $C \subset \bar{V} \subset U$ and

- (i) $F_0 = f$
- (ii) $F_t|_{M-U} = f|_{M-U}$
- (iii) $F_1|_V = h|_V$

Proof. Let $\mathcal{O} \subset N \times N$ be an invariant neighborhood of the diagonal in $N \times N$ such that for all $(x, y) \in \mathcal{O}$ there is a unique minimal geodesic $\gamma_t(x, y)$ with $\gamma_0(x, y) = x$, $\gamma_1(x, y) = y$. Define $H: U \rightarrow N \times N$ by $H(\eta) = (f(\eta), h(\eta))$. Let $U' = H^{-1}(\mathcal{O})$ and choose an open set V in M so that $V \subset U'$. Let $\lambda: M \rightarrow [0, 1]$ be invariant and differentiable with $\lambda(M-U) = 0$ and $\lambda(V) = 1$ and define

$$F_t(\eta) = \begin{cases} \gamma_{\lambda(\eta)t}(f(\eta), h(\eta)) & \eta \in U' \\ f(\eta) & \eta \in M - U' \end{cases}$$

Clearly F_t has the desired properties.

Let $\pi: E \rightarrow B$ be a Riemannian G -vector bundle. Then there is a canonical decomposition $T(E)|_B \approx T(B) \oplus E$. If $\pi': E' \rightarrow B'$ is another differentiable G -vector bundle and $f: E \rightarrow E'$ is a differentiable equivariant map preserving the zero-section, define $\widetilde{df}: E \rightarrow E'$ by the composition $E \rightarrow T(B) \oplus E \approx T(E)|_B \xrightarrow{df} T(E')|_{B'} \approx T(B') \oplus E' \rightarrow E'$, \widetilde{df} is a bundle homomorphism, the linearization of f . f is said to be linear on $E(m)$ if $f|_{E(m)} = \widetilde{df}|_{E(m)}$.

LEMMA 3.5. Let f be as above with B compact and suppose f linear on $(E|_C)(\eta)$ where C is closed in B . Then there is a differentiable equivariant homotopy F_t of f such that

- (i) $F_0 = f$
- (ii) F_1 is linear on $E(\delta)$ for some $\delta > 0$
- (iii) $F_t|_{E-E(2\delta)} = f|_{E-E(2\delta)}$
- (iv) $F_t|_{(E|_C)} = f|_{(E|_C)}$

Proof. Apply Lemma 3.4 with $h = \widetilde{df}$, $U = E(2\delta)$; (iv) follows by choosing $2\delta < \eta$.

Let $V \subset W$ be orthogonal representations of G and let M be a compact G -manifold equivariantly imbedded in the representation space V with $p: \nu(M) \rightarrow M$ the normal bundle

of this imbedding. Let $\pi: G_k(W) \rightarrow G_{r-k}(W)$ be the equivariant diffeomorphism defined by $D(y) = Id - y$; here $r = \text{dimension } W$ and a point $y \in G_k(W)$ is regarded as an orthogonal projection: $W \rightarrow W$ with nullity k , $D(y)$ clearly is an orthogonal projection with nullity $r - k$.

Definition. Let $M \subset V$ and $p: v(M) \rightarrow M$; an equivariant bundle *epimorphism* $f: v(M) \rightarrow \mu_k(W)$ is said to be consistent (with respect to the inclusion of V in W) at $x \in M$ if the following diagram is commutative:

$$\begin{array}{ccc} W & \xrightarrow{D(f(x))} & W \\ \cup & & \cup \\ V & & \\ \cup & & \\ p^{-1}(x)^{G_x} & \xrightarrow{f} & \pi^{-1}(f(x))^{G_x}. \end{array}$$

The symbol U^H denotes the orthogonal complement of the fixed point set in the representation space U of the group H , i.e. $U^H = (U_H)^\perp$. f is said to be consistent on $C \subset M$ if f is consistent at each $x \in C$.

PROPOSITION 3.6. *Let $N^{n+1} \subset V^{n+h} \oplus \mathbf{R}$ and let $f: v(N)|U \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ ($k + n + 1 = \dim(V^{n+h} \oplus \mathbf{R})$) be a consistent bundle map where U is a neighborhood of the closed invariant set $C \subset U \subset N$. Then $f|(v(N)|C)$ may be extended to a consistent bundle map $v(N) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$.*

COROLLARY 3.7. *Let $M^n \subset V^{n+h}$ and let $f_i: v(M) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ $i = 1, 2$, be consistent bundle maps. Then there is a homotopy $F: v(M) \times [0, 5] \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ such that*

- (i) $F_0 = f_1$
- (ii) $F_5 = f_2$
- (iii) F_t is a consistent bundle map for each t .

Proof. Apply the above theorem to $M \times [0, 5] \subset V^{n+h} \oplus \mathbf{R}$, $U = M \times [0, 1) \cup M \times (4, 5]$, $C = M \times 0 \cup M \times 5$ and $f: v(M) \times [0, 5] \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ defined by

$$f(v, t) = \begin{cases} f_1(v) & t < 1 \\ f_2(v) & t > 4 \end{cases} \quad v \in v(M)$$

Remark. The corollary may be paraphrased, $\pi: \mu_k(V^{n+h} \oplus \mathbf{R}) \rightarrow G_k(V^{n+h} \oplus \mathbf{R})$ is $(n + 1)$ universal for consistent bundle maps.

Proof of Proposition. By Lemma 2.2 we must find a non-singular equivariant section of the G -vector bundle $\text{Hom}(v(N), V^{n+h} \oplus \mathbf{R})$ which extends the section, s_f over C defined by f . A section, s , is said to be consistent at x if

$$\begin{array}{ccc} p^{-1}(x) & \xrightarrow{s(x)} & V^{n+h} \oplus \mathbf{R} \\ \cup & & \cup \\ p^{-1}(x)^{G_x} & \xrightarrow{id} & p^{-1}(x)^{G_x} \end{array}$$

is commutative, i.e. if $s(x)|p^{-1}(x)^{G_x}$ is the identity. Note that a consistent non-singular section defines a consistent bundle map and vice-versa. For $A \subset B \subset N$, $H \subset G$, let $\Gamma_H(B, A)$ denote the consistent H equivariant sections of $\text{Hom}(v(N), V^{n+h} \oplus \mathbf{R})$ over $B \cup C$ which extend the section s_f and are non-singular on A . Note that $\Gamma_G(N, \emptyset)$ is a closed subset of the

space of equivariant sections of $\text{Hom}(v(N), V^{n+h} \oplus \mathbf{R})$ with the $C-0$ topology) and hence is a complete metric space. We shall prove

- (i) $\Gamma_G(N, \emptyset) \neq \emptyset$
- (ii) for each $x \in N - C$, there is a compact invariant set C_x such that $x \in C_x$, $C_x \cap C = \emptyset$ and there exists a countable number of C_{x_i} such that $UC_{x_i} \cup C = N$
- (iii) $\zeta: \Gamma_G(N, \emptyset) \rightarrow \Gamma_G(C_x, \emptyset)$ is open
- (iv) $\Gamma_G(C_x, C_x)$ is open and dense in $\Gamma_G(C_x, \emptyset)$.

Then by (iii) and (iv) $\Gamma_G(N, C_x)$ is open and dense in $\Gamma_G(N, \emptyset)$; by (ii) and Baire's theorem $\cap \Gamma_G(N, C_{x_i}) = \Gamma_G(N, N)$ is open and dense in $\Gamma_G(N, \emptyset)$ and hence by (i) there exists a non-singular equivariant consistent section and thus a consistent bundle map $v(N) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ extending f .

- (i) let s_f be the section over U defined by f and let s_N be the section over N defined by $v(N) \subset T(V^{n+h} \oplus \mathbf{R})|N = (V^{n+h} \oplus \mathbf{R}) \times N \rightarrow V^{n+h} \oplus \mathbf{R}$. Let $\lambda: N \rightarrow I$ be differentiable and invariant with $\lambda(N - U) = 0$, $\lambda(C) = 1$. Then $s(x) = \lambda(x)s_f(x) + (1 - \lambda(x))s_N(x)$ is clearly consistent hence $\Gamma_G(N, \emptyset) \neq \emptyset$.
- (ii) for each $x \in N - C$, there is a slice S_x in N such that $S_x \cap C = \emptyset$; then let $C_x = G((S_x)_{G_x})$, i.e. $C_x = \{y \in GS_x | G_y \text{ is conjugate to } G_x\}$. If $P \subset V^{n+h} \oplus \mathbf{R}$ is a submanifold (not necessarily compact) then P may be covered by a countable number of C_{x_i} . Note that if there is only one orbit type in P , i.e. all G_x , $x \in P$, are conjugate then C_x contains a neighborhood of x and hence a countable number of C_{x_i} will cover P . If there are r orbit types in P , let (H) be the minimal orbit type, i.e. $H = G_x$ for some $x \in P$ and there does not exist a $y \in P$ with $G_y \supset H$; then $P_0 = \{x \in P | G_x \text{ is conjugate to } H\}$ is a closed submanifold of P with only one orbit type and hence can be covered by a countable number of C_{x_i} (it is immaterial whether one chooses a slice in P_0 or a slice in P to define C_x). Moreover, $P - P_0$ has only $r - 1$ orbit types and hence by induction may be covered by a countable number of C_{x_i} ; therefore P may be so covered.
- (iii) to show that $\zeta: \Gamma_G(N, \emptyset) \rightarrow \Gamma_G(C_x, \emptyset)$ is open, it is sufficient to show that if $s \in \Gamma_G(N, \emptyset)$ and $s' \in \Gamma_G(C_x, \emptyset)$ with $\|s' - s|C_x\| < \varepsilon$ then there exists a $s'' \in \Gamma_G(N, \emptyset)$ with $\zeta(s'') = s'|C_x = s'$ and $\|s'' - s\| < 3\varepsilon/2$. Suppose that s' can be extended to a consistent section s''' in a neighborhood U of C_x ; then since $\|s'''|C_x - s|C_x\| < \varepsilon$ there exists a neighborhood V of C_x with $\|s|V - s'''|V\| < 3/2\varepsilon$.

Let $\lambda: N \rightarrow I$ be invariant and differentiable with $\lambda(N - V) = 0$, $\lambda(C_x) = 1$ and let $s'''(x) = \lambda(x)s'''(x) + (1 - \lambda(x))s(x)$ then s''' clearly has the desired property.

To establish the neighborhood extension property for consistent sections and the set C_x we first note that $\Gamma_G(C_x, \emptyset) = \Gamma_{G_x}(S_x)_{G_x}, \emptyset)$ by equivariance. Moreover, $S_x(2)$ (the slice of radius 2 at x) is equivariantly contractible and hence by Corollary 2.6 $v(M)|S_x(2) \simeq S_x(2) \times W \times \mathbf{R}^a$ where W is a representation space of G_x and $k = a + \dim W$. Let $\theta: v(M)|S_x(2) \rightarrow S_x(2) \times W \times \mathbf{R}^a$ be an equivalence. Via θ an element $s \in \Gamma_{G_x}(S_x(2), \emptyset)$ may be regarded as a pair of G_x equivariant maps $s_1: S_x(2) \rightarrow \text{Hom}(W, V^{n+h} \oplus \mathbf{R})$, $s_2: S_x(2) \rightarrow \text{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$. If $s' \in \Gamma_{G_x}((S_x)_{G_x}, \emptyset)$ then $s'_2: (S_x)_{G_x} \rightarrow \text{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$ may clearly be extended to a map $s'_2: S_x(2) \rightarrow \text{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$ since $(S_x)_{G_x}$ is a G_x equivariant retract of $S_x(2)$. To

show that s'_1 may be extended so as to be consistent we note that s'_1 is defined on $(S_x)_{G_x}$ by $s'_1(y): y \times W \times 0 \subset (S_x)_{G_x} \times W \times \mathbf{R}^a \xrightarrow{\theta} \nu(N)|(S_x)_{G_x} \subset T(V^{n+h} \oplus \mathbf{R})|(S_x)_{G_x} \approx (S_x)_{G_x} \times V^{n+h} \oplus \mathbf{R} \rightarrow V^{n+h} \oplus \mathbf{R}$ and hence s'_1 may be extended to $s'_1: S_x(2) \rightarrow \text{Hom}(W, V^{n+h} \oplus \mathbf{R})$ by $s'_1: S_x(2) \times W \times 0 \subset S_x(2) \times W \times \mathbf{R}^a \approx \nu(M)|S_x(2) \subset S_x(2) \times V^{n+h} \oplus \mathbf{R} \rightarrow V^{n+h} \oplus \mathbf{R}$. Hence the section $s''(y) = s'_1(y) + s'_2(y)$ defined by s'_1 and s'_2 clearly extends s' and is consistent. Thus $\Gamma_{G_x}(S_x(2), \emptyset) \rightarrow \Gamma_{G_x}((S_x)_{G_x}, \emptyset)$ and hence $\Gamma_G(GS_x(2), \emptyset) \rightarrow \Gamma_G(C_x, \emptyset)$ is onto:

- (iv) to show that $\Gamma_G(C_x, C_x)$ is open and dense in $\Gamma_G(C_x, \emptyset)$ or equivalently that $\Gamma_{G_x}((S_x)_{G_x}, (S_x)_{G_x})$ is open and dense in $\Gamma_{G_x}(S_x)_{G_x}, \emptyset)$ let $s \in \Gamma_{G_x}((S_x)_{G_x}, \emptyset)$ $s = s_1 + s_2$ as before where $s_1: (S_x)_{G_x} \rightarrow \text{Hom}(W, V^{n+h} \oplus \mathbf{R})$, $s_2: (S_x)_{G_x} \rightarrow \text{Hom}(\mathbf{R}^a, V^{n+h} \oplus \mathbf{R})$. Note that $s_1(y)$ is a monomorphism for each y by consistency, hence it is sufficient to show that s_2 can be approximated by a map s'_2 with $s'_2(y)$ a monomorphism for each y . Since G_x acts trivially on $(S_x)_{G_x}$ and s_2 is G_x equivariant, we may regard s_2 as a map into $\text{Hom}(\mathbf{R}^a, (V^{n+h} \oplus \mathbf{R})_{G_x})$.

Letting $F_j = \{T \in \text{Hom}(\mathbf{R}^a, (V^{n+h} \oplus \mathbf{R})_{G_x}) | \text{rank } T = j\}$ $j = 0, 1, \dots, a-1$, we see that codimension $F_j < \dim(S_x)_{G_x}$ (Lemma 2.3) since $\dim(S_x)_{G_x} + \dim \mathbf{R}^a + \dim T(G_x)_{G_x} = \dim(V^{n+h} \oplus \mathbf{R})_{G_x}$ and hence $\dim(S_x)_{G_x} + a \leq \dim(V^{n+h} \oplus \mathbf{R})_{G_x}$. Thus, s_2 may be approximated arbitrarily closely by a map transversal to $F_0 \cup F_1 \dots \cup F_{a-1}$, i.e. by a map s'_2 with $s'_2(y)$ a monomorphism for each y . Then $s' = s_1 + s'_2$ is a non-singular approximation showing that $\Gamma_G(C_x, C_x)$ is dense in $\Gamma_G(C_x, \emptyset)$. Clearly $\Gamma_G(C_x, C_x)$ is open.

Definition. Let $W \subset V^{n+h} \oplus \mathbf{R}$ and let $f: W \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ be a differentiable equivariant map. Then f is said to be consistently transverse regular (CTR) at $0 \in W$ if $f(0) \notin G_k(V^{n+h} \oplus \mathbf{R})$ or if $f(0) \in G_k(V^{n+h} \oplus \mathbf{R})$ then

- (i) $f|W_G: W_G \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})_G$ is transverse regular to $G_k(V^{n+h} \oplus \mathbf{R})_G$ at 0, and if $F = (f|W_G)^{-1}(G_k(V^{n+h} \oplus \mathbf{R})_G)$ then
- (ii) f is locally linear at F and
- (iii) $f: \nu(F) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ is consistent.

f is said to be CTR at $w \in W$ if $f|S_w$ is CTR as a G_w map where S_w is the slice at w defined by the end point map. f is said to be CTR on $C \subset W$ if f is CTR at each $x \in C$.

LEMMA 3.8. *If f is CTR on a neighborhood of $W_G(1)$ in W_G then there is a neighborhood of $W_G(1)$ in W on which f is CTR.*

Proof. Follows immediately from local linearity.

LEMMA 3.9. *Let $f: W \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ be CTR in a neighborhood U of the closed set C . If $(V^{n+h} \oplus \mathbf{R})^G \subset W$ then there is a homotopy $F_t: W \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ such that*

- (i) $F_0 = f$
- (ii) $F_t|W - W(2) = f|W - W(2)$
- (iii) $F_t|C = f|C$
- (iv) F_1 is CTR on a neighborhood of $C \cup W_G(1)$.

Proof. By Lemma 3.3 we may assume that $f|W_G$ is TR to $G_k(V^{n+h} \oplus \mathbf{R})_G$ in $\mu_k(V^{n+h} \oplus \mathbf{R})_G$ at points of $W_G(2)$. Let $F = (f|W_G(2))^{-1}(G_k(V^{n+h} \oplus \mathbf{R})_G)$. Then by Lemma 3.5 we may assume that $f|W_G(2)$ is linear on $\nu(F, W_G)(\delta)$ for some $\delta > 0$. There is at most one CTR map

$h: v(F)(\delta) \cup U \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ such that $h|U = f$ and $h|v(F, W_G)(\delta) = f$ and, since $W \supset (V^{n+h} \oplus \mathbf{R})^G$ exactly one; hence, by Lemma 3.4 there is a homotopy F_t satisfying (i), (ii) and (iii) with $F_1|v(F, W_G)(\delta) = h$, i.e. with F_1 satisfying (iv).

Let V^{n+h} be identified with $S(V^{n+h} \oplus \mathbf{R}) - \{\text{north pole}\}$ in some fixed way; it then makes sense to talk of a map $f: (S(V^{n+h} \oplus \mathbf{R})) \rightarrow (T_k V^{n+h} \oplus \mathbf{R}, \infty)$ being CTR.

LEMMA 3.10. *Let $X = S(V^{n+h} \oplus \mathbf{R})$ or $S(V^{n+h} \oplus \mathbf{R}) \times I$ and let $f: X \rightarrow T_k(V^{n+h} \oplus \mathbf{R})$ be an equivariant differentiable map which is CTR on a neighborhood of the closed set $C \subset X$. If G_x acts trivially on $T(G/G_x)_e$ for each $x \in X$, then f is homotopic to a map \bar{f} which is CTR on X . Moreover, \bar{f} may be chosen so that $\bar{f}|C = f|C$.*

Proof. Note that if $x \in X$, the G_x space S_x satisfies the hypothesis of Lemma 3.9, $S_x \supset (V^{n+h} \oplus \mathbf{R})^{G_x}$, since G_x acts trivially on $T(G/G_x)_e$; and $S_x + T(G/G_x)_e = V^{n+h} \oplus \mathbf{R}|_{G_x}$.

If H is an isotropy group in X , define the level of H by level $G = 0$; level $H \geq s$ if $H \subsetneq H'$ where H' is an isotropy group with level $H' = s - 1$; level $H = s$ if level $H \geq s$ and level $H \not\geq s + 1$. Let $X_r = \{x \in X | \text{level } G_x \leq r\}$. Then $X_{-1} = \emptyset$ and $X_0 = X_G$. Suppose that $X = X_s$ and that $f_r: X \rightarrow T_k(V^{n+h} \oplus \mathbf{R})$ is defined so that

- (i) f_r is homotopic to f ,
- (ii) $f_r|C = f|C$,
- (iii) f_r is CTR on U_r where U_r is an open neighborhood of $C \cup X_r$.

If $f_{-1} = f$ then (i), (ii), (iii) above are satisfied and hence we proceed by induction. Since $X_{r+1} - (U_r \cap X_{r+1})$ is compact we may choose a finite number of slices $S_{x_i}(3)$, $i = 1, \dots, m$, $x_i \in X_{r+1} - (U_r \cap X_{r+1})$ such that $\bigcup_{i=1}^m GSx_i$ covers $X_{r+1} - (U_r \cap X_{r+1})$ and $GSx_i(3) \cap (C \cup X_r = \emptyset$. Let $f_{r+1}^{-1} = f_r$ and suppose inductively that f_{r+1}^l has been defined so that

- (i) f_{r+1}^l is homotopic to f ,
- (ii) $f_{r+1}^l|C = f|C$,
- (iii) f_{r+1}^l is CTR on $U_r - G\left(\bigcup_{i=1}^l S_{x_i}(3)\right)$,
- (iv) f_{r+1}^l is CTR on $G\left(\bigcup_{i=1}^l (S_{x_i})_{G_{x_i}}\right)$:

Applying Lemma 3.9 to $f_{r+1}^l|S_{x_{l+1}}(3)$ and the closed subset $\hat{S}_{x_{l+1}}(3) \cap G\left(\bigcup_{i=1}^l (S_{x_i})_{G_{x_i}}\right)$ we get a map f_{r+1}^{l+1} such that (i) to (iv) are satisfied with $l + 1$ replacing l . Finally let $f_{r+1} = f_{r+1}^m$. Then f_{r+1} is homotopic to f and $f_{r+1}|C = f|C$ by construction. Moreover, f_{r+1} is CTR on $U_r - G\left(\bigcup_{i=1}^m S_{x_i}(3)\right) \supset C$. Since f_{r+1} is also CTR on X_{r+1} , by Lemma 3.8 there is a neighborhood U_{r+1} of $C \cup X_{r+1}$ on which f_{r+1} is CTR. Hence, the inductive hypothesis is satisfied and $\bar{f} = f_s$ has the required properties.

Note that if G is finite or if G is abelian then G_x acts trivially on $T(G/G_x)_e$ and hence lemma 3.10 holds.

THEOREM 3.11. *If G is finite or abelian then $\theta: \mathfrak{n}_n(V) \rightarrow \pi_1^{V^{n+h}}(T_k(V^{n+h} \oplus \mathbf{R}), \infty)$ is an isomorphism.*

Proof. θ is onto : let $[f] \in \pi_1^{V^{n+h}} T_k(V^{n+h} \oplus \mathbf{R})$. By Lemma 3.10 there is a CTR map \tilde{f} with $[\tilde{f}] = [f]$. Let $\tilde{f}^{-1}(G_k(V^{n+h} \oplus \mathbf{R})) = M$. Then $\theta([M]) = [f_{M,i}] = [\tilde{f}]$ since the bundle maps $v(M) \rightarrow \mu_k(V^{n+h} \oplus \mathbf{R})$ defined by \tilde{f} and $f_{M,i}$ are consistent and, therefore, homotopic by Corollary 3.7; the Thom construction applied to $M \times I \subset S(V^{n+h} \oplus \mathbf{R}) \times I$ then yields a homotopy between \tilde{f} and $f_{M,i}$. Hence θ is onto.

To show that θ is a monomorphism suppose $\theta([M]) = 0$, i.e. suppose $f_{M,i}$ is equivariantly homotopic to $[0]$. If we knew that $f_{M,i}$ was a CTR map Lemma 3.10 would imply that there was a CTR homotopy $F: S(V^{n+h} \oplus \mathbf{R}) \times I \rightarrow T_k(V^{n+h} \oplus \mathbf{R})$ with $F_0 = f_{M,i}$ and $F_1 = [0]$ and hence $F^{-1}(G_k(V^{n+h} \oplus \mathbf{R}))$ would provide a cobordism between M and \emptyset , i.e. would show that $[M] = 0$. The only difficulty is that $f_{M,i}$ need not be locally linear.

Let $i: M \subset V^{n+h}$ and let $x \in M$. Then the G_x space, $T(M)_x$, splits as the direct sum of $T(Gx)_x$, the tangent space to the orbit and its orthogonal complement W (orthogonal with respect to the metric on M induced by i). Recall that any slice S_x is the image of a G_x equivariant diffeomorphism $\psi: W(\varepsilon) \rightarrow M$; $\psi(W(\varepsilon)) = S_x$.

Definition. The imbedding $i: M \rightarrow V^{n+h}$ is said to be straight at $x \in M$ if, for some slice S_x at x , $S_x = \psi(W(\varepsilon))$, there is a $\delta > 0$ such that the map $\psi': W(\delta) \rightarrow V^{n+h}$ given by $W(\delta) \subset W_{G_x}(\delta) \times W^{G_x}(\delta) \xrightarrow{\psi'} V^{n+h}$, $\psi'(y, z) = i \circ \psi(y) + di_{\psi(y)}(z)$ for $y \in W_{G_x}(\delta)$; $z \in W^{G_x}(\delta)$ defines a slice at $i(x) \in i(M)$, i.e., $\psi'(W(\delta)) \subset i(M)$ and $i^{-1} \circ \psi': W(\delta) \rightarrow M$ defines a slice at $x \in M$.

Remark 1. It is clear that this condition is independent at the particular slice S_x or map ψ .

Remark 2. If i is straight at x , then i is straight on a neighborhood of x , in fact, on $G(\psi'(W)(\delta))$.

Remark 3. The map $f_{M,i}$ is CTR in a neighborhood of x if and only if i is straight at x . Hence to complete the proof of Theorem 3.11 we need only show there exists an imbedding $i: M \rightarrow V^{n+h}$ such that i is straight at each $x \in M$.

LEMMA 3.12. *Let $i: M \rightarrow V^{n+h}$ be an imbedding which is straight on a neighborhood U at the closed invariant set C . Let $S_x(2)$ be a slice of radius 2 at $x \in M$. Then there is an imbedding $\bar{i}: M \rightarrow V^{n+h}$ such that $\bar{i}|_C = i|_C$ and \bar{i} is straight on $C \cup G((S_x)_{G_x})$.*

Proof. Let $\psi: W(\varepsilon) \rightarrow S_x(2)$ be as above and define $h: S_x(2) \rightarrow V^{n+h}$ by the composition. $S_x(2) \xrightarrow{\psi^{-1}} W(\varepsilon) \subset W_{G_x}(\varepsilon) \times W^{G_x}(\varepsilon) \xrightarrow{\psi'} V^{n+h}$ where $\psi'(y, z) = i \circ \psi(y) + di_{\psi(y)}(z)$ for $y \in W_{G_x}(\varepsilon)$ $z \in W^{G_x}(\varepsilon)$ and extend h to $GS_x(2)$ by equivariance. Let $\lambda: M \rightarrow [0, 1]$ be an invariant differentiable map with $\lambda(C \cup M - GS_x(2)) = 0$, $\lambda(S_x(1) - U) = 1$. Let $i_1: M \rightarrow V^{n+h}$ be defined by $i_1(p) = (1 - \lambda(p))i(p) + \lambda(p)h(p)$. Note that $i_1|_{C \cup G(S_x)_{G_x}} = i|_{C \cup G(S_x)_{G_x}}$ and $di_1|_{C \cup G(S_x)_{G_x}} = di|_{C \cup G(S_x)_{G_x}}$ and hence that i_1 is an imbedding of a closed neighborhood Q of $C \cup G(S_x)_{G_x}$. By construction $i_1|_Q$ is straight on $C \cup (S_x)_{G_x}$. Let $\bar{i}: M \rightarrow V^{n+h}$ be an imbedding with $\bar{i}|_Q = i_1|_Q$ (Corollary 1.10). Then \bar{i} satisfies the stated conditions.

Remark. Note that the metric induced from V^{n+h} by \bar{i} and that induced by i agree on $C \cup G(S_x)_{G_x}$ and hence \bar{i} is straight on C since i was straight on C .

To show that M admits a straight imbedding in V^{n+h} one proceeds by induction on the level sets $M_r = \{x \in M \mid \text{level } G_x \leq r\}$ as in Lemma 3.10. Lemma 3.12 justifies the inductive step.

§4. EQUIVARIANT MORSE THEORY

In this section we extend the results of R. Palais in [14] to study an invariant C^∞ function $f: M \rightarrow \mathbf{R}$ on a complete Riemannian G -space M .

Definition. At a critical point p of f , i.e., where $\Delta f_p = 0$, we have a bounded, self-adjoint operator, the hessian operator, $\varphi(f)_p = T(M)_p \rightarrow T(M)_p$ defined by $\langle \varphi(f)_p v, w \rangle = H(f)_p(v, w)$ where $H(f)_p$ is the hessian bilinear form [14, §7]. A closed invariant submanifold V of M will be called a *critical manifold* of f if $\partial V = \emptyset$, $V \cap \partial M = \emptyset$ and if each $p \in V$ is a critical point of f . It follows that $T(V)_p \subseteq \ker \varphi(f)_p$ and so there is an induced bounded self-adjoint operator $\bar{\varphi}(f)_p: T(M)_p/T(V)_p \rightarrow T(M)_p/T(V)_p$. If $\bar{\varphi}(f)_p$ is an isomorphism for each $p \in V$ then V is called a *non-degenerate critical manifold* of f .

Recall that f is said to satisfy condition (C) [14, §10] if, for each closed subset S of M on which f is bounded, $\|\Delta f\|$ is bounded away from zero or there is a critical point $p \in S$.

Definition. The invariant C_∞ function $f: M \rightarrow \mathbf{R}$ is called a *Morse function* for the Riemannian G -manifold M if it satisfies condition (C) and if the critical locus of f is a union of non-degenerate critical manifolds without interior.

The behavior of a function near a critical manifold is specified by the Morse Lemma.

LEMMA 4.1. *Let $\pi: E \rightarrow B$ be a Riemannian G -vector bundle and f a Morse function on E having B (i.e., the zero section) as a non-degenerate critical manifold. If B is compact there is an equivariant diffeomorphism $\theta: E(r) \rightarrow E$ for some $r > 0$ such that $f(\theta(e)) = \|Pe\|^2 - \|(1-P)e\|^2$ where P is an equivariant orthogonal bundle projection.*

Proof. Let $E_x = \pi^{-1}(x)$ and let $i_x: E_x \rightarrow E$, $p_x: T(E)_x \rightarrow T(E)_x/T(B)_x$ then from the commutative diagram we see that

$$\begin{array}{ccc}
 T(E_x)_0 & \xrightarrow{di_x} & T(E)_x \\
 \downarrow \varphi(f \circ i_x)_0 & \searrow p_x & \downarrow \varphi(f)_x \\
 & T(E)_x/T(B)_x & \\
 & \downarrow \bar{\varphi}(f)_x & \\
 & T(E)_x/T(B)_x & \\
 & \swarrow p_x & \\
 T(E_x)_0 & \xrightarrow{di_x} & T(E)_x
 \end{array}$$

$\varphi(f \circ i_x)_0$ is an isomorphism. Hence, in each fibre, 0 is a non-degenerate critical point of the function $f \circ i_x$ and hence, by the results of [12], there is an origin preserving diffeomorphism $\theta_x: E_x \rightarrow E_x$ and a projection P_x such that $f \circ i_x \circ \theta_x(e) = \|P_x(e)\|^2 - \|(1 - P_x)(e)\|^2$ in a neighborhood of the origin. To complete the proof, we must show that θ_x and P_x are smooth functions of x and that the resulting maps $\theta: E \rightarrow E$, $P: E \rightarrow E$ are equivariant.

Let $\text{Hom}(E, E)$ denote the G -vector bundle over B with fibre $\text{Hom}(E_x, E_x)$ at x where $\text{Hom}(E_x, E_x)$ denotes the bounded linear operators on E_x and the action of G on $\text{Hom}(E, E)$ is given by $gT = \bar{g} \cdot T \cdot \bar{g}^{-1}$ where $T \in \text{Hom}(E_x, E_x)$ and $gT \in \text{Hom}(E_{gx}, E_{gx})$. We regard $B \subset E$ via the zero section. We shall define an equivariant fibre preserving map $A: E \rightarrow \text{Hom}(E, E)$ such that

- (i) $A(e)$ is a self-adjoint operator for each $e \in E$,
- (ii) $f(e) = \langle A(e)e, e \rangle$,
- (iii) if $x \in B$, $\bar{\varphi}(f)_x \circ p_x \circ di_x = 2p_x \circ di_x \circ A(x)$

$$\begin{array}{ccccccc} E_x = T(E_x)_0 & \xrightarrow{di_x} & T(E)_x & \xrightarrow{p_x} & T(E)_x / T(B)_x \\ \downarrow 2\Lambda(x) & & \downarrow \varphi(f \cdot i_x)_0 & & \downarrow \varphi(f)_x & & \downarrow \bar{\varphi}(f)_x \\ E_x = T(E_x) & \xrightarrow{di_x} & T(E)_x & \xrightarrow{p_x} & T(E)_x / T(B)_x \end{array}$$

A is given by

$$\langle A(e)v_1v_2 \rangle = \int_0^1 (1-t) d^2(f \circ \psi^{-1})_{\varphi(te)}(d\psi_{te}(\bar{v}_1), d\psi_{te}(\bar{v}_2)) dt,$$

where $\psi: \pi^{-1}(U) \rightarrow U \times F$ is any bundle chart for E at $\pi(e)$ and \bar{v}_i denotes the tangent vector at te corresponding to $v_i \in E$, i.e., $\bar{v}_i = (di_{\pi(e)})_{te}(v_i)$. Property (i) follows from the symmetry of $d^2(f \circ \psi^{-1})$ and (iii) follows from the fact that $tx = x$ for $x \in B$ and $\int_0^1 (1-t) dt = 1/2$.

Since $f(B) = 0$ and $df|_B = 0$ Taylor's formula for f with $n = 1$ yields the remainder term

$$f(e) = \int_0^1 (1-t) d^2(f \circ \psi^{-1})_{\psi(te)}(d\psi_{te}(\bar{e}), d\psi_{te}(\bar{e})) dt = \langle A(e)e, e \rangle$$

and hence (ii). To show that A is well-defined we apply the chain rule to $(f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}) = f \circ \psi^{-1}$ where $\varphi: \pi^{-1}(U) \rightarrow U \times F$ is another bundle chart, noting that $\varphi \circ \psi^{-1}$ is linear in each fibre and hence $d^2(\varphi \circ \psi^{-1})_e(\bar{v}_1, \bar{v}_2) = 0$ for $v_1, v_2 \in E$. Then

$$\begin{aligned} d^2(f \circ \psi^{-1})_{\psi(e)}(d\psi_e(\bar{v}_1), d\psi_e(\bar{v}_2)) &= d^2(f \circ \varphi^{-1})_{\varphi(e)}(d\varphi_e(\bar{v}_1), d\varphi_e(\bar{v}_2)) \\ &\quad + d(f \circ \varphi^{-1})_{\varphi(e)}[d^2(\varphi \circ \psi^{-1})_{\psi(e)}(d\psi_e(\bar{v}_1), d\psi_e(\bar{v}_2))] \\ &= d^2(f \circ \varphi^{-1})_{\varphi(e)}(d\varphi_e(\bar{v}_1), d\varphi_e(\bar{v}_2)) \end{aligned}$$

and hence A is well-defined. To demonstrate the equivariance of A we note that if ψ is a bundle chart at $\pi(e)$ then

$$(\bar{g} \times id) \circ \psi \circ \bar{g}^{-1}: \pi^{-1}(gU) \xrightarrow{g^{-1}} \pi^{-1}(U) \longrightarrow U \times F \xrightarrow{g \times id} gU \times F$$

is a bundle chart at $\pi(ge)$. Then $\langle A(ge)gv_1, gv_2 \rangle \stackrel{\text{def}}{=} d^2(f \circ \bar{g} \circ \psi^{-1} \circ (\bar{g}^{-1} \times id))_{(\bar{g} \times id)\psi(e)}(d(\bar{g} \times id)d\psi_e(\bar{v}_1), d(\bar{g} \times id)d\psi_e(\bar{v}_2))$ and by the invariance of f and the chain rule this equals

$$d^2(f \circ \psi^{-1})_{\psi(e)}(d\psi_e(\bar{v}_1), d\psi_e(\bar{v}_2)) + d(f \circ \psi^{-1})_{\psi(e)}(d^2(\bar{g}^{-1} \times id)(d\bar{g}(\bar{v}_1), d\bar{g}(\bar{v}_2)).$$

Since $\bar{g}^{-1} \times id$ is linear in each fibre $d^2(\bar{g}^{-1} \times id) = 0$ and hence $\langle A(ge)gv_1, gv_2 \rangle = \langle A(e)v_1, v_2 \rangle$ and thus $A(ge) = g \circ A(e) \circ g^{-1}$ since the metric is invariant. The maps θ, P are limits of polynomials in A and hence are equivariant and differentiable. The rest of the proof follows formally as in [14, §7].

An important property of Morse functions is given by:

PROPOSITION 4.2. *If f is a Morse function the critical locus of f in $f^{a,b} = f^{-1}[a, b]$ is the union of a finite number of disjoint, compact, non-degenerate critical manifolds of f .*

Proof. Let $\{a_n\}$ be a sequence at points in the critical set. Since, by assumption, a_n is in a non-degenerate critical manifold without interior we may choose points $\{b_n\}$ such that

- (i) the distance $\rho(a_n, b_n) < \frac{1}{n}$
- (ii) $a - 1 < f(b_n) < b + 1$
- (iii) $0 < \|\Delta f_{b_n}\| < \frac{1}{n}$.

Then by condition (C) there is a critical point p adherent to $\{b_n\}$ and hence $\{b_n\}$ has a subsequence which converges to p . The corresponding subsequence of $\{a_n\}$ will also converge to p , thus proving the compactness of the critical set in $f^{a,b}$.

We also have the Diffeomorphism Theorem.

THEOREM 4.3. *Let f be a Morse function on M , $\partial M = \emptyset$, with no critical value in the bounded interval $[a, b]$. If $f^{a-\delta, b+\delta}$ is complete for some $\delta > 0$ then $f^a = f^{-1}(-\infty, a]$ is equivariantly diffeomorphic to f^b .*

Proof. Essentially, this theorem is Proposition 2, Section 10 of [14]. We need only verify that the map defined there is equivariant. The map is given by $p \rightarrow \sigma_p(\alpha(f(p)))$ where $\alpha: \mathbf{R} \rightarrow \mathbf{R}$ is C_∞ ; hence

$$gp \rightarrow \sigma_{gp}(\alpha(f(gp))) = \sigma_{gp}(\alpha(f(p))) = g\sigma_p(\alpha(f(p))).$$

COROLLARY 4.4. (Palais and Stewart [13]). *Every differentiable deformation ψ_t of a G -manifold M is trivial.*

Proof. Recall that a differentiable deformation is a one-parameter family of actions $\psi_t: G \times M \rightarrow M$ such that the action $\psi: G \times M \times \mathbf{R} \rightarrow M \times \mathbf{R}$ given by $\psi(g, m, t) = (\psi_t(g, m), t)$ is differentiable. ψ_t is trivial if there is a one-parameter family of diffeomorphisms θ_t of M such that $\psi_t(g, m) = \theta_t \psi_0(g, \theta_t^{-1}(m))$. Let $M \times \mathbf{R}$ have a complete invariant metric with respect to ψ and let $f: M \times \mathbf{R} \rightarrow \mathbf{R}$ be the projection onto the second factor. Since f is a Morse function and has no critical points, the map $\theta_t(p) = \sigma_p(t)$ has the required properties.

Definition. Let V, W be Riemannian G -vector bundles over B . The bundle $V(1) \oplus W(1) = \{(x, y) \in V \oplus W \mid \|x\| \leq 1, \|y\| \leq 1\}$ (not a manifold) is called a handle-bundle of type (V, W) with index = dimension of W . Let N, M be G -manifolds with boundary, $N \subset M$ and $\bar{F}: V(1) \oplus W(1) \rightarrow M$ a homeomorphism onto a closed subset H of M . Let $F = \bar{F}|_{V(1) \oplus W(1)}$. We shall write $M = N \cup_F H$ and say that M arises from N by attaching a handle-bundle of type (V, W) if

- (i) $M = N \cup H$
- (ii) F is an equivariant diffeomorphism onto $H \cap \partial N$
- (iii) $\bar{F}|_{V(1) \oplus W(1)}$ is an equivariant diffeomorphism onto $M - N$.

LEMMA 4.5 (Attaching Lemma). *Let $\pi: E \rightarrow B$ be a Riemannian G -vector bundle and P an orthogonal bundle projection. Let $V = P(E)$, $W = (1 - P)(E)$ and define $f, g: E \rightarrow \mathbf{R}$ by $f(e) = \|Pe\|^2 - \|(1 - P)e\|^2$, $g(e) = f(e) - 3\varepsilon/2\lambda(\|Pe\|^2/\varepsilon)$ where $\varepsilon > 0$ and λ is a positive C^∞ function which is monotone decreasing, $\lambda([0, 1/2]) = 1$ and $\lambda(1) = 0$. Then $\{x \in E(2\varepsilon) | g(x) \leq -\varepsilon\}$ arises from $\{x \in E(2\varepsilon) | f(x) \leq -\varepsilon\}$ by attaching a handle-bundle of type (V, W) .*

Proof. Let $\sigma(s)$ be the unique solution of $\lambda(\sigma)/1 + \sigma = 2/3(1 - s)$ for $s \in [0, 1]$. Define $\bar{F}: V(1) \oplus W(1) \rightarrow E$ by $\bar{F}(x, y) = (\varepsilon\sigma(\|x\|^2)\|y\|^2 + \varepsilon)^{1/2}x + (\varepsilon\sigma(\|x\|^2))^{1/2}y$. It is shown in Section 11 of [14] that \bar{F} has the required properties.

Note that B is a non-degenerate critical manifold of f . By the Morse lemma we can choose coordinates for $\pi: E \rightarrow B$ and a projection P such that $f(e) = \|Pe\|^2 - \|(1 - P)e\|^2$ in a neighborhood of B for any function f having B as a non-degenerate critical manifold. Hence, by abuse of notation, we shall also refer to the handle-bundle of type $(P(E), (1 - P)E)$ as a handle-bundle of type (B, f) .

THEOREM 4.6. *Let f be a Morse function on the complete Riemannian G -space M . If f has a single critical value $a < c < b$ in the bounded interval $[a, b]$ then the critical locus of f in $[a, b]$ is the disjoint union of a finite number of compact submanifolds N_1, \dots, N_s . f^b is equivariantly diffeomorphic to f^a with s handle-bundles of type (N_i, f) disjointly attached.*

Proof. Only the last statement remains. Let $\{U_i\}_{i=1, \dots, s}$ be disjoint tubular neighborhoods of the critical submanifolds $\{N_i\}$ given by the maps $T_i: \nu(N_i)(2\delta) \rightarrow U_i$ where $\nu(N_i)$ is the normal bundle of N_i in M with the induced Riemannian metric. We may assume $c = 0$ and by the Morse Lemma that $f \circ T_i(x) = \|P_i x\|^2 - \|(1 - P_i)x\|^2$ where P_i is an orthogonal bundle projection in $\nu(N_i)$. Choose ε so that $0 < \varepsilon < \delta^2$ and $a < -3\varepsilon$, $3\varepsilon < b$.

Let $Q = f^{-2\varepsilon, \infty}$ and define $g: Q \rightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & x \notin \bigcup_{i=1}^s U_i \\ f(x) - 3\varepsilon/2\lambda(\|P_i T_i^{-1}(x)\|^2/\varepsilon) & x \in U_i \end{cases}$$

where λ is the function defined in the Attaching Lemma. It is shown in (14, §11) that g is C^∞ and $g^\varepsilon = (f|Q)^\varepsilon$. Moreover, by the Attaching Lemma, $g^{-\varepsilon}$ is equivariantly diffeomorphic to $(f|Q)^{-\varepsilon} \cup_s$ handle-bundles of type (N_i, f) . Since f has no critical value in $[a, -\varepsilon]$ or $[\varepsilon, b]$ it is sufficient to show that $g^{-\varepsilon} \approx g^\varepsilon$. To that end we apply the Diffeomorphism Theorem to the manifold without boundary $g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$ and the function g . We note that $g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$ is complete and hence we need only show that g is a Morse function, i.e., $\|\nabla g\|$ is bounded away from zero for $x \in g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$. Since $g(N_i) = -3\varepsilon/2$, $N_i \cap g^{-1}(-5\varepsilon/4, 5\varepsilon/4) = \emptyset$. Hence there is an $\alpha > 0$ such that $T_i(\nu(N_i)(\alpha)) \cap g^{-1}(-5\varepsilon/4, 5\varepsilon/4) = \emptyset$. Moreover, $f(g^{-1}(-5\varepsilon/4, 5\varepsilon/4)) \subset [-5\varepsilon/4, 5\varepsilon/4]$ and hence, since f has no critical points in $g^{-1}[-5\varepsilon/4, 5\varepsilon/4]$, $\|\nabla f_x\|$ must be bounded away from zero, say $\|\nabla f_x\| \geq \eta > 0$. But

$$g|Q - \bigcup_{i=1}^s U_i = f|Q - \bigcup_{i=1}^s U_i \quad \text{and hence} \quad \|\nabla g_x\| \geq \eta > 0 \quad \text{for} \quad x \in Q - \bigcup_{i=1}^s U_i.$$

Thus we need only show that $\|\nabla g\| | U_i \cap g^{-1}(-5\varepsilon/4, 5\varepsilon/4)$ is bounded away from zero. To compute $\|\nabla g\|$ we first construct a Riemannian metric \langle, \rangle^* for $T(\nu(N_i))$ such that $\langle \bar{v}_1, v_2 \rangle =$

$\langle v_1, v_2 \rangle$ where $v_i \in v(N_i)$, \langle, \rangle denotes the metric in $v(N_i)$ and \bar{v}_i denotes the tangent vector at $x \in v(N_1)$ corresponding to v_i . Then if $x \in v(N_i)$, let $\bar{w} = P_i(x) - (1 - P_i)(x) \in T(v(N_i))_x$. We have $d(g \circ T)_x(\bar{w}) = 2[\langle Px, \bar{w} \rangle - \langle 1 - Px, \bar{w} \rangle] - 3\lambda'(\|Px\|^2/\varepsilon)\langle Px, \bar{w} \rangle$ since $g \circ T(x) = \|Px\|^2 - \|1 - Px\|^2 - 3\varepsilon/2\lambda(\|Px\|^2/\varepsilon)$. Since $\lambda'(t) \leq 0$, $d(g \circ T)_x(\bar{w}) = 2\|x\|^2 - 3\lambda'(\|Px\|^2) \geq 2\|x\|^2$. But $d(g \circ T)_x(\bar{w}) = dg_{Tx}(dT_x(\bar{w})) = \langle \nabla g_{Tx}, dT_x(\bar{w}) \rangle \leq \|\nabla g_{Tx}\| \|dT_x(\bar{w})\| \leq \|\nabla g_{Tx}\| \|dT_x\| \|\bar{w}\| = \|\nabla g_{Tx}\| \|dT_x\| \|x\|$. Since $\|x\| \geq \alpha$ we see that $\|\nabla g_{Tx}\| \geq \|2\alpha/\|dT_x\|\|$. We need only show that $\|dT_x\|$ is bounded. Since N_i is compact $\|dT\|$ is bounded on $N_i \subset v(N_i)$ and hence in a neighborhood $v(N_i)(\beta)$ of N_i . Hence since δ was arbitrary we assume $2\delta < \beta$. Finally, we have $(f|Q)^b \approx (f|Q)^e = g^e \approx g^{-e} \approx (f|Q)^{-e} \cup s$ handle-bundles of type (N_i, f) and therefore $f^b \approx f^{-e} \cup s$ handle-bundles $\approx f^a \cup s$ handle-bundles. The homology implications of the above theorem are contained in

COROLLARY 4.7 (Bott [2]). *Let N_1, \dots, N_r be those critical manifolds in $f^{a,b}$ with index $(N_i, f) = k_i < \infty$. Then*

$$H_n(f^b, f^a; Z_2) \approx \sum_{i=1}^r H_{n-k_i}(N_i; Z_2).$$

Proof. By the above theorem $f^b \approx f^a \cup s$ handle-bundles of type (N_i, f) . Let $H_i = V_i(1) \oplus W_i(1)$ denote the i th handle-bundle and let $P_i: V_i \oplus W_i \rightarrow V_i \oplus W_i$ denote the projection onto V_i . Then by excising out the interior of f^a we have

$$H_n(f^b, f^a; Z_2) \approx \sum_{i=1}^s H_n(H_i, V_i(1) \oplus \dot{W}_i(1); Z_2).$$

But $H_n(H, V(1) \oplus \dot{W}(1); Z_2) = H_n(W(1), \dot{W}(1); Z_2)$ since the fibre of H is convex and we have an equivariant fibre preserving retraction, ρ , of H onto $V(1) \oplus \dot{W}(1) \cup 0 \oplus W(1)$ given by

$$\begin{aligned} \rho(h) &= \rho(P(h), (1 - P)(h)) = \rho(x, y) \\ &= \begin{cases} \left(\frac{2x}{2 - \|y\|}, 0 \right) & \text{if } \|x\| \leq 1 - \frac{\|y\|}{2} \\ \left(\frac{x}{\|x\|}, (2\|x\| + \|y\| - 2) \frac{y}{\|y\|} \right) & \text{if } \|x\| \geq 1 - \frac{\|y\|}{2}. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} H_n(f^b, f^a; Z_2) &\approx \sum_{i=1}^s H_n(W_i(1), \dot{W}_i(1); Z_2) \\ &\approx \sum_{i=1}^r H_{n-k_i}(N_i; Z_2) + \sum_{i=r+1}^s H_n(W_i(1), \dot{W}_i(1); Z_2) \end{aligned}$$

where the last isomorphism is the Thom isomorphism for $i \leq r$. It only remains to show that $H_n(W(1), \dot{W}(1); Z_2) = 0$ if $\dim W = \infty$ or even strong that $\pi_m(W(1), \dot{W}(1)) = 0$ for all m . Let $\alpha: D^n, S^{n-1} \rightarrow W(1), \dot{W}(1)$ represent an element of $\pi_m(W(1), \dot{W}(1))$. We may approximate α by a map α' which is homotopic to α , differentiable and transverse regular to N , the zero section. Since codimension $N = \infty$, $\alpha'(D^n) \cap N = \emptyset$ and we can deform α' into $\dot{W}(1)$ and hence $[\alpha'] = 0$. Thus critical manifolds of infinite index do not affect the homology of (f^b, f^a) .

Now let a, b be arbitrary regular values of f , $a < b$, and again denote the critical manifolds of finite index k_i by $\{N_i\}$, $i = 1, \dots, t$. Let $R_n(X) = \text{dimension of } H_n(X; \mathbb{Z}_2)$ and $\chi(X)$ the Euler characteristic of X . Then we have the Morse inequalities.

$$(i) \quad \chi(f^b, f^a) = \sum_{i=1}^t (-1)^{k_i} \chi(N_i)$$

$$(ii) \quad R_n(f^b, f^a) \leq \sum_{i=1}^t R_{n-k_i}(N_i)$$

$$(iii) \quad \sum_{i=0}^n (-1)^{n-i} R_i(f^b, f^a) \leq \sum_{i=1}^t \sum_{l=0}^n (-1)^{n-l} R_{l-k_i}(N_i).$$

The statements follow from the above corollary and the fact that χ is additive, and $R_n, \sum_{n \leq k} (-1)^{k-n} R_n$, are subadditive ([14], §15).

Remark. If every critical manifold of finite index in $f^{a,b}$ has an orientable normal bundle then equations (i), (ii), (iii), are valid with integer coefficients.

We now show that there exist Morse functions on any finite-dimensional G -manifold, M . To that end let $\mathcal{M}_G(A, M) \subset C_G(M, \mathbb{R})$ denote those functions whose critical locus in A is a union of non-degenerate critical orbits. Clearly $\mathcal{M}_G(A, M)$ is open if A is compact.

DENSITY LEMMA 4.8. *For any finite-dimensional G -manifold M , $\mathcal{M}_G(M, M)$ is dense in $C_G(M, \mathbb{R})$.*

Proof. Let $x \in M - M_G$. By the induction metatheorem of [13] we may assume that $\mathcal{M}_{G_x}(S(x), S(x))$ is dense in $C_{G_x}(S(x), \mathbb{R})$, where $S(x)$ is a slice at x . Since the restriction map $\rho: C_G(M, \mathbb{R}) \rightarrow C_{G_x}(S(x), \mathbb{R})$ is open, $\rho^{-1}(\mathcal{M}_{G_x}(S(x), S(x))) = \mathcal{M}_G(B(x), M)$ is dense in $C_G(M, \mathbb{R})$. Now let $y \in M_G$ and let $A = B_y \cap M_G$. We show that $\mathcal{M}_G(A, M)$ is dense in $C_G(M, \mathbb{R})$ and then complete the proof with Baire's theorem. Let $f: M \rightarrow \mathbb{R}$. We must find a C^k approximation, f' , such that f' has only non-degenerate critical points in A . We note that $\mathcal{M}(A, M_G)$ is dense in $C(M_G, \mathbb{R})$ (10, p. 37] and that the restriction map $C_G(M, \mathbb{R}) \rightarrow C(M_G, \mathbb{R})$ is open. Hence, we may assume that $f|_{M_G}$ has only non-degenerate critical points and by induction that y is the only critical point in A which is degenerate for f (y is non-degenerate for $f|_{M_G}$). This problem is local and is settled by the following.

LEMMA 4.9. *Let W be an Euclidean G -space and $f: W \rightarrow \mathbb{R}$ an invariant C^∞ function such that $f|_{W_G}$ has only non-degenerate critical points and such that $0 \in W$ is the only degenerate critical point of f in $W(1)$. Then there exists a C^∞ invariant function $f': W \rightarrow \mathbb{R}$ such that*

$$(i) \quad f'|_{W - W(2)} = f|_{W - W(2)}$$

$$(ii) \quad f' \text{ has only non-degenerate critical points in } W_G(1)$$

$$(ii) \quad f' \text{ is a } C^k \text{ approximation to } f.$$

Proof. Let $P: W \rightarrow W$ denote the internal projection onto W_G . Define f' by $f'(w) = f(w) + \varepsilon \lambda(\|w/c\|^2) \|(1 - P)w\|^2$, where ε, c are constants to be chosen and λ is the function of Lemma 4.5. We choose $c < 2$ such that if $x \in W_G$ is a critical point of f , then $\|x\| > c$ or $x = 0$; this is clearly possible since $f|_{W_G}$ has only isolated critical points by the Morse Lemma. Then note that $f'|_{W_G} = f|_{W_G}$ and $f'|_{W - W(c)} = f|_{W - W(c)}$ which proves (i)

and shows that f' has at most 0 as a degenerate critical point. By definition of f' , $\varphi_0(f')(v) = \varphi_0(f)(v) + 2\varepsilon(1 - P)v$ or in matrix form

$$\varphi_0(f') = \begin{bmatrix} \varphi_0(f|W_G) & C \\ D & B + 2\varepsilon I \end{bmatrix}$$

where B , C , D are determined by f and φ_0 is the Hessian operator. But $\varphi_0(f|W_G)$ is non-singular since $f|W_G$ has only non-degenerate critical points and hence $\det \varphi_0(f')$ is a non-zero polynomial in ε with roots $\varepsilon_1, \dots, \varepsilon_n$; (iii) can then be satisfied by choosing ε small enough and (ii) be demanding that $\varepsilon \neq \varepsilon_i$.

Remark. Let $f \in \mathcal{M}_G(C, M)$ where C is closed and $\varepsilon: M \rightarrow \mathbf{R}$ a positive function. Let $C_G(f, C, \varepsilon) = \{h \in C_G(M) | h|C = f|C \text{ and } |h(x) - f(x)| < \varepsilon(x)\}$. Then $C_G(f, C, \varepsilon)$ is of the second category and the same argument as above shows that $\mathcal{M}_G(M, M) \cap C_G(f, C, \varepsilon)$ is dense in $C_G(f, C, \varepsilon)$.

COROLLARY 4.10. *There exists a Morse function on M .*

Proof. Let $\{\psi_i\}$ be a countable partition of unity with compact support. Then $f(x) = \sum_{i=1}^{\infty} i\psi_i(x)$ is proper. Uniformly approximating f by a function in $C_G(f, \varphi, 1) \cap \mathcal{M}_G(M, M)$ yields a Morse function.

COROLLARY 4.11. *If M is compact then M is equivariantly diffeomorphic to $(N_1, f) \cup_{g_2}(N_2, f) \dots \cup_{g_k}(N_k, f)$ where the (N_i, f) are handle-bundles over orbits. M has the equivariant homotopy type of $(V_1(1) \times_{H_1} G) \cup_{g_2}(V_2(1) \times_{H_2} G) \dots \cup_{g_n}(V_n(1) \times_{H_n} G)$ where $V_i(1) \times_H G$ is a disc bundle over G/H_i and the g_i are attaching maps.*

Proof. Let $f \in \mathcal{M}_G(M, M)$ and apply the main theorem to f and the interval $[\min f - 1, \max f + 1]$ to get the first statement. The second follows from the deformation defined in Corollary 4.7.

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